Etude des taux d'interet long terme Analyse stochastique des processus ponctuels determinantaux

Camilier Isabelle

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Isabelle Camilier

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Etude des taux d'intérêt long terme Analyse stochastique des processus ponctuels déterminantaux

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Laurent Decreusefond	Directeur de thèse
Laurent Denis	Examinateur
Nicole El Karoui	Directrice de thèse
Alfred Galichon	Examinateur
Emmanuel Gobet	Président du jury
Eckhard Platen	Rapporteur
Nicolas Privault	Rapporteur
Marie-Claire Quenez	Rapporteur

Résumé

Dans la première partie de cette thèse, nous donnons un point de vue financier sur l'étude des taux d'intérêt long terme. En finance, les modèles classiques de taux ne s'appliquent plus pour des maturités longues (15 ans et plus). En nous inspirant de travaux d'Economie, mais en tenant compte de l'existence d'un marché financier, nous montrons que les techniques de maximisation d'utilité espérée permettent de retrouver la règle de Ramsey (qui relie la courbe des taux à l'utilité marginale de la consommation optimale). En marché incomplet, il est possible de montrer un analogue de la règle de Ramsey et nous examinons la manière dont la courbe des taux est modifiée. Ensuite nous considérons le cas où il y a une incertitude sur un paramètre du modèle, puis nous étendons ces résultats au cas où les fonctions d'utilités sont stochastiques. Alors la courbe des taux dépend de la richesse de l'économie.
D'autre part nous proposons dans cette thèse une nouvelle manière d'appréhender la consommation, comme des provisions que l'investisseur met de côté pour les utiliser en cas d'un évènement de défaut. Alors le problème de maximisationn de l'utilité espérée de la richesse et de la consommation peut être vu comme un problème de maximisation de l'utilité espérée de la richesse terminale avec un horizon aléatoire.

La deuxième partie de cette thèse concerne l'analyse stochastique des processus ponctuels déterminantaux. Les processus déterminantaux et permanentaux sont des processus ponctuels dont les fonctions de corrélations sont données par un déterminant ou un permanent. Les points de ces processus ont respectivement un comportement de répulsion ou d'attraction: ils sont très loin de la situation d'absence de corrélation rencontrée pour les processus de Poisson. Nous établissons un résultat de quasi-invariance: nous montrons que si nous perturbons les point le long d'un champ de vecteurs, le processus qui en résulte est toujours un déterminantal, dont la loi est absolument continue par rapport à la distribution d'origine. En se basant sur cette formule et en suivant l'approche de Bismut du calcul de Malliavin, nous donnons ensuite une formule d'intégration par parties.

Abstract

The first part of this thesis concerns a financial point of view of the study of long term interest rates. We seek an alternative to classical interest rates models for longer maturities (15 years and more). Our work is inspired by the work of economists, but takes into account the existence of a (complete) financial market. We show that classical expected utility maximization techniques lead to the Ramsey Rule, linking the yield curve and marginal utility from consumption. We extend the Ramsey Rule to the case of an incomplete financial market and examine how the yield curve is modified. It is then possible to consider the case where there is incertainty on a parameter of the model, then to extend these results to the case of dynamic utility functions, where the yield curve depends on level of wealth in the economy.
The other main result we present is a new way of considering the consumption, as a quantity of supplies that the investor puts aside and uses in case of a default event. Then the expected utility maximization from consumption and terminal wealth can be interpreted as a problem of maximization of expected utility from terminal wealth with a random horizon.

The topic of the second part of this thesis is the stochastic analysis of determinantal point processes. Determinantal and permanental processes are point processes with a correlation function given by a determinant or a permanent. Their atoms exhibit mutual attraction or repulsion, thus these processes are very far from the uncorrelated situation encountered in Poisson models. We establish a quasi-invariance result : we show that if atoms locations are perturbed along a vector field, the resulting process is still a determinantal (respectively permanental) process, the law of which is absolutely continuous with respect to the original distribution. Based on this formula, following Bismut approach of Malliavin calculus, we then give an integration by parts formula.

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Contents

Long term interest rates and market numeraire: a financial point of view
Stochastic analysis of determinantal point processes

There are two parts in this thesis. The first part of this work has been carried out under the supervision of Prof. Nicole El Karoui and concerns a financial point of view of the study of long-term interest rates (Chapters 1 to 5). The second one, under the supervision of Prof. Laurent Decreusefond, concerns the stochastic analysis of determinantal point processes. Both parts are self-contained and can be read independently.

The first part of this thesis concerns a financial point of view of the modelling of long term interest rates. For the financing of ecological projects reducing global warming, longevity questions or any other investment with a long term impact, it is necessary to model accurately long term interest rates. But for longer maturities (20 years and more), the interest rate market becomes highly illiquid and standard interest rates models cannot be easily extended. There is however an abundant economic literature on long-term policy-making. In the first chapter, we present the point of view of the economists on this subject. It is based on a representative agent maximizing his utility from consumption. The derivation of the yield curve for far-distant maturities is induced from the maximization of the representative agent's utility function from consumption. The formula linking the yield curve and the consumption of the agent is called the Ramsey Rule. Then we discuss the various extensions of the Ramsey Rule, and the economic assumptions it is based on.
Our work is a financial point of view on long term interest rates: it is inspired by the economic literature on long-term policy-making but we take into account the existence of a financial market. This is the purpose of Chapter 2. A first way to achieve this is to consider a complete market where an agent maximizes the expected utility of his consumption and terminal wealth under a budget constraint. Our contribution is to examine the classical utility maximization techniques from the point of view of interest rates. We show that classical expected utility maximization techniques lead again to the Ramsey Rule. Then we underline the role of the Growth Optional Portfolio, which has been studied in detail by Heath and Platen [PH06]. We refer to his work in Chapters 2 and 3. The Growth Optimal Portfolio is a particularly robust portfolio over long periods of time and therefore it is a useful tool for the study long term interest rates. The Growth Optimal portfolio can be used as a numeraire for pricing zero coupons.

The expressions of zero-coupon bond prices or of the yield curve hold for a finite horizon T. We choose $T \leq T^H$, we choose a finite horizon. For the moment, in this work we have not considered the case where $T \to +\infty$. In Economics, this case is often mentioned. But in finance, the density of the risk-neutral probability tends to zero when $T \to +\infty$. But throughout this work, we remain in the case of a long term but finite horizon.
A the end of this Chapter we explain how it is possible to link yield curve dynamics with historical data.

In the third Chapter, our framework is extended to the case of an incomplete financial market (where incompleteness comes from portfolio contraints). Then the pricing probability is not universal and might depend on the maturity and the utility function. We examine the consequences of incompleteness on the term structure of long term interest rates. Of course in this case, an important issue is how to price zero-coupons in this framework? It is shown that the Ramsey Rule holds if we adopt a pricing rule linked to the marginal utility (Davis prices). In this Chapter, we present a dual formulation of the expected utility maximization problem more suitable. We underline the fact that the optimal dual process depends on y, the wealth in the economy.

The most important results of this part are in Chapters 4 and 5.
Various extensions and new results are given in Chapter 4. In particular, consider the case where there is uncertainty on a parameter of the model. This hypothesis makes sense for investment problems with a long term horizon. What would be the impact on the yield curve? Another extension is the case where a particular agent has more information on a parameter of the model. This question can be treated using filtration enlargement techniques.

But the most important result of this chapter is a new point of view on consumption. We interpret the consumption process as a certain quantity of wealth, or supplies. In this framework, the agent non longer invests and consumes. Instead, he agent invests in the financial market and makes supplies. He will use these supplies only if an unpredictable event (or default) happens before maturity. The mathematical formulation of these ideas introduces a new market which we call the $\mathbb{G}$-market, with new utility functions, which are stochastic. These new results involve progessive filtration enlargement, this is why we treat them in this chapter. These results are also used in the next chapter. Thus, this chapter proposes a new point of view of the consumption process, which is a key quantity in the expression of the yield curve via the Ramsey Rule.

In Chapter 5, another main contribution of this part is presented: we extend the results obtained to the case of dynamic utility functions. Indeed, so far we have not taken into account the fact that the representative agent could change his preferences during the observed time period, meaning that the agent utility function could change over time. This would be especially true in the case of long term investments. In order to take this into account, we use progressive consistent dynamic utility functions, introduced by Musiela and Zariphopoulo for utility functions from terminal wealth, and developed by El Karoui and Mrad. First, we extend the definition of dynamic utility functions to dynamic utility from consumption functions. Then, using Chapter 4, we define dynamic utility functions in the $\mathbb{G}$-market. Once again, in this chapter, the dual formulation is particularly important. Then, using results from the previous chapters, the last step is to study the yield curve. It is important to notice that its dynamics depends on the level of wealth in the economy.

The topic of the second part of this thesis (Chapters 6 and 7) is the stochastic analysis of determinantal point processes. Determinantal and permanental processes are point processes with a correlation function given by a determinant or a permanent. Their atoms exhibit mutual attraction or repulsion, thus these processes are very far from the uncorrelated situation encountered in Poisson models. However, a part of our work is inspired by the method of Albeverio et al. for establishing integration by parts formulas for Poisson measures.
We establish first a quasi-invariance result : we show that if atoms locations are perturbed along a vector field, the resulting process is still a determinantal (respectively permanental) process, the law of which is absolutely continuous with respect to the original distribution. Based on this formula, following Bismut approach of Malliavin calculus, we give an integration by parts formula. It is then possible to generalize this formula for a larger family of point processes called alpha-determinantal processes (where the parameter α measures the strength of the repusion between points). Then we study a method for the simulation of determinantal point processes in $\mathbb{R}^n$, based on an acception-rejection method and different from the one of Hough et al.

Part I

Long-term interest rates: a financial point of view

Introduction

Les taux d'intérêt long terme

La première partie de mon travail de thèse concerne l'étude des taux d'intérêt long terme. Ce problème est un enjeu majeur, pour estimer le coût du financement des questions liés aux problèmes écologiques, ou au problème du vieillissement de la population.
En effet, au cours des dernières années, dans le domaine du développement durable, beaucoup de questions se sont posées concernant le financement de projets à long terme, c'est à dire à un horizon temporel de $T = 50$ ans et plus. Ces questions sont motivées par le financement au niveau mondial de projets écologiques et environnementaux. Un exemple typique est la question du nombre de points de croissance que nous devons sacrifier aujourd'hui pour réduire les effets du réchauffement climatique.
D'autre part, il existe des contrats liés au risque de mortalité, qui ont typiquement une maturité de 20 ans. Il existe aussi des produits faisant intervenir le risque de longévité, qui sont de maturité plus longue (40 ans et au-delà), voir [BBK$^+$09]. Ces contrats sont sensibles au risque de taux.
Ainsi, lorsqu'on s'intéresse à toutes ces questions, la modélisation des taux d'intérêt devient inévitable. Pour des maturités qui ne sont pas trop longues (jusqu'à 20 ans), les modèles de taux standart en finance peuvent être utilisés, le marché des taux étant relativement liquide pour ces maturités.

Les définitions suivantes concernant l'approche classique des taux d'intérêt se trouvent dans les travaux de Brigo et Mercurio [BF01], Musiela et Rutkowski [MR00], Heath, Jarrow et Morton [HJM98] ou El Karoui [ElK]. Un zéro-coupon de maturité T est un titre qui donne à celui qui le détient une unité de cash à la date T. Le prix à la date t d'un zéro-coupon de maturité T telle que $0 \leq t \leq T$ sera noté $B(t,T)$. On appelle rendement à l'échéance (ou yield to maturity) en t la fonction $Y_t(T)$ telle que: $B(t,T) = \exp(-(T-t)Y_t(T))$. La structure par terme des taux d'intérêt ou courbe des taux est la fonction qui associe à la maturité θ le taux $R_T(\theta)$:

$$\theta \to R_T(\theta) = Y_T(T+\theta) = -\frac{1}{\theta}\ln B(T, T+\theta),$$

Pour des maturités relativement courtes, des modèles de taux peuvent être utilisés, par exemple des modèles de taux courts, qui permettent ensuite de déduire la forme de la courbe des taux. Considérons par exemple le modèle de taux suivant (modèle de Vasicek), en marché complet. Ce modèle suppose que les taux courts sont donnés par:

$$dr_t = a(b - r_t)dt - \sigma dW_t^{\mathbb{Q}},$$

où $W^{\mathbb{Q}}$ est un Brownien sous la probabilité risque-neutre. A partir d'un modèle de taux court, il est possible d'en déduire les autres caractéristiques de la courbe des taux: le prix des zéros-coupons $B(0,T) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\big(-\int_0^T r_s ds\big)\Big]$, qui s'exprime sous la probabilité risque-neutre. Enfin la courbe des taux entre la date 0 et la date T est donnée par: $R_0(T) = -\frac{1}{T}\ln B(0,T)$.

Cependant pour des maturités plus longues, le marché des taux d'intérêt devient très illiquide. Le point de vue standart sur les taux d'intérêt ne peut être étendu facilement et il faut envisager une autre approche.
Une possibilité est de s'inspirer de la littérature économique. En effet, en Economie, une littérature abondante concernant les aspects économiques des politiques à long terme (c'est à dire à un horizon temporel de 50 à 200 ans) a été développée. Nous nous inspirons de la théorie économique de Gollier ([Gol]), Scheinkman ([HS09]) et Breeden ([Bre89]). Dans cette approche l'économie est représentée par la stratégie d'un agent représentatif, considéré comme averse au risque avec une fonction d'utilité $u(.)$ dérivable, croissante et concave. On appelle β son paramètre de préférence pour le présent, c'est à dire que β quantifie la préférence de biens consommés immédiatement par rapport à ceux consommés dans le futur.
Dans ce cadre, le taux d'intérêt est donné par la formule qui donne la consommation en fonction de l'utilité optimale à l'équilibre. La courbe des taux vus d'aujourd'hui pour des maturités longues $R(0,T)$ est déduite de la maximisation de l'utilité intertemporelle de la consommation c_t de l'agent représentatif, en prenant en compte la contrainte de budget.

$$\max_{c \geq 0} \int_{t \geq 0} e^{-\beta t} u(c_t) dt$$

Dans ce cadre non aléatoire, les taux et la consommation sont déterministes, et la consommation optimale est donnée par:

$$u'(\widehat{c}_t)e^{-\beta t} = u'(c_0)e^{-\int_0^t r_s ds}$$

où r_t est le taux court et le terme $e^{-\int_0^t r_s ds}$ est le facteur d'actualisation. Dans ce cadre non aléatoire, la courbe des taux s'écrit comme une moyenne des taux courts. On obtient ainsi la règle de Ramsey:

$$R(t) = \frac{1}{t}\int_0^t r_s ds = \beta - \frac{1}{t}\ln\frac{u'(\widehat{c}_t)}{u'(c_0)}.$$

En particulier, avec des hypothèses simplificatrices sur la fonction d'utilité ou la trajectoire de la consommation, il est possible d'obtenir la formule suivante pour la règle de Ramsey:

$$R(t) = \beta + \frac{1}{\eta}g, \tag{0.0.1}$$

où β est le taux de préférence pour le présent, η est l'aversion au risque de l'agent et g est l'anticipation sur la croissance économique. Récemment, dans le cadre de l'étude du taux d'escompte à appliquer pour le financement de projets à long terme, la règle de Ramsey a été discutée par de nombreux auteurs parmi lesquels Gollier [Gol, Gol07b, Gol09c, Gol07a, Gol08, Gol09b, Gol06, Gol09a], Weitzman [Wei98, Wei07], Stern (dans le Stern Review of Climate Change [Ste]), Ekeland [Eke], Jouini et al. [JN10, JMN10]. La question du choix des paramètres dans la règle de Ramsey même sous sa forme simplifiée a été source de nombreuses controverses parmi les économistes. Dans le premier chapitre de cette thèse, nous présentons ces différentes approches. Suivant les paramètres choisis, on obtient différentes valeurs du taux $R(t)$. Par exemple, le rapport Stern préconise un paramètre de préférence pour le présent de $\beta = 0.1\%$, une aversion pour le risque $\eta = 1$ et un taux de croissante $g = 1.3\%$, soit un taux constant $R(t) = 1.4\%$. Beaucoup d'économistes trouvent ce taux trop bas et proposent le choix de paramètres suivant: $\beta = 2\%, \eta = 1/2, g = 2\%$, c'est à dire un taux de $R(t) = 6\%$. Suivant l'une ou l'autre des valeurs des taux, la valeur dans 50 ans d'une certaine somme d'argent ne sera pas du tout la même. Cela montre a quel point la question du choix des paramètres est cruciale. Cette divergence dans les anticipations des paramètres n'est pas étonnante: il est déjà difficile de prévoir les paramètres de l'économie (par exemple la croissance) une an à l'avance, cela est d'autant plus vrai sur une durée de plusieurs dizaines d'années. Les travaux de Jouini et al. [JN10, JMN10] proposent de réconcilier ces divergences dans les anticipations des agents. Dans son modèle, un certain nombre d'agents ont chacun des anticipations différentes sur le taux de croissance de l'économie g_i, et n'ont pas non plus la même préférence pour le présent. Chacun d'entre eux croit à un taux $R_i(t)$. Un équilibre s'établit. Alors pour des courtes maturités, la taux $R(t)$ est une moyenne pondérée des taux $R_i(t)$ des différents agents. Puis la courbe des taux décroit, et pour des maturités longues $R(t) = R_{i_0}(t)$: à long terme c'est le taux de l'agent le plus pessimiste qui est choisi. C'est le taux le plus bas, il est donc proche de zéro. Le fait de choisir un taux très bas pour des maturités longues a aussi donné lieu à des controverses que nous évoquerons.

Par ailleurs, jusque là, l'approche des économistes ne prend pas en compte l'existence d'un marché financier. Dans notre cas, pour étudier des problèmes liées aux taux d'intérêt long terme, nous avons besoin de mettre le marché financier au coeur de nos préoccupations. Une première façon d'aboutir dans le cas d'un marché liquide est de considérer un agent représentatif qui optimise l'utilité espérée de sa consommation sous une contrainte de budget (c'est le problème d'investissement consommation en marché complet).

$$\max_{c\geq 0} \int_{t\geq 0} e^{-\beta t}\mathbb{E}\left(u(\widetilde{c}_t)\right) dt \quad \text{s.t. } \mathbb{E}^{\mathbb{Q}}\left(\int_{t\geq 0} e^{-\int_0^t \widetilde{r}_s ds}\widetilde{c}_t dt\right) \leq x_0,$$

où dans la contrainte de budget on évalue la valeur actualisée des flux futurs sous la probabilité de pricing risque-neutre $\mathbb{Q}$. Alors en notant Z_t^0 la densité de $\mathbb{Q}$ par rapport à la probabilité historique $\mathbb{P}$, nous avons la relation suivante:

$$\exp(-\beta t)u'(c_t^*) = u'(c_0^*)\exp\left(-\int_0^t \widetilde{r}_s ds\right) Z_t^0.$$

Alors, en notant $B(0,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_0^T \tilde{r}_s ds)\right]$ le prix à la date 0 d'un zéro-coupon de maturité T, on obtient le lien suivant entre le prix d'un zéro-coupon et l'utilité marginale de la consommation optimale:

$$B(0,T) = \exp(-\beta T)\mathbb{E}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right].$$

Sachant que $B(0,T) = \exp(-R_0(T)T)$, nous montrons qu'on peut retrouver la règle de Ramsey, d'un point de vue financier dans le cadre d'un marché complet. C'est à dire :

$$R_0(T) = \beta - \frac{1}{T}\ln \mathbb{E}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right]. \tag{0.0.2}$$

Plan Cette partie de la thèse est constituée de plusieurs chapitres. Dans un premier chapitre, nous présentons le point de vue des économistes. Nous donnons la règle de Ramsey, ou Ramsey Rule, traduisant le lien entre consommation et courbe des taux. Nous présentons les hypothèses sur lesquelles elle repose, la signification des différents paramètres, et nous discutons des divergences entre économistes au niveau du choix des paramètres.
Notre objectif est de donner un point de vue financier aux questions de long terme, c'est ce que nous faisons à partir du Chapitre 2. Pour cela nous utilisons deux outils: la maxmisation d'utilité et le Growth Optimal Portfolio ou numéraire de marché (le portefeuille optimal pour l'utilité logarithmique). Nous considérons le cas d'un marché complet, où un agent représentatif maximise l'utilité espérée de sa richesse terminale et de sa consommation. Cette théorie est maintenant bien connue, mais nous nous intéressons aux problèmes de taux dans ce contexte. En marché complet, la courbe des taux est donnée par le marché. Mais il est possible d'établir la Ramsey Rule, qui fait le lien entre la courbe des taux et l'utilité de la consommation optimale. Par ailleurs, si on s'affranchit temporairement de l'hypothèse de l'existence d'une probabilité risque-neutre, il est cependant possible d'avoir des informations sur la courbe des taux. Cela se fait grâce au au Growth Optimal Portfolio, étudié en détail par Heath et Platen, qui donne une vision historiquement testable de la tendance générale des marchés.
Le Chapitre 3 concerne le marché incomplet. Ce chapitre peut être vu comme un prolongement du précédent. L'hypothèse d'un marché complet étant assez restrictive, une idée assez naturelle est d'étendre le cadre précédent au cas du marché incomplet et d'examiner comment la courbe des taux est modifiée dans ce cas. C'est en marché incomplet que la Ramsey Rule prend toute sa force.
Cette approche peut également être enrichie en prenant en compte le cas où il y a une incertitude sur l'un des paramètres, alors la consommation initiale est elle-même stochastique.
Pour notre étude du long terme il est aussi crucial de tenir compte du fait que l'agent peut changer ses préférences au cours de la période de temps observée, c'est à dire que la fonction d'utilité de l'agent varie du cours du temps. Pour rendre compte de cet effet, nous étendons nos résultats aux utilités dynamiques, toujours avec un point de vue centré sur les taux d'intérêt.

Maximisation d'utilité, taux d'intérêt et portefeuille optimal

Les chapitres 2 et 3 sont un retour sur les problèmes d'optimisation d'utilité. Dans le chapitre 2, nous présentons successivement un agent économique averse au risque qui peut investir dans un marché financier complet, les préférences de l'agent, puis son problème de maximisation d'utilité espérée. Nous revisitons ensuite les questions de courbes de taux dans ce contexte.
L'agent représentatif qui investit dans ce marché financier choisit entre un actif sans risque au taux d'intérêt $r \geq 0$ et N actifs risqués (actions). Etant donné un horizon d'investissement fini T, et une richesse initiale $x > 0$, on note par le vecteur π son portefeuille, qui représente la richesse investie dans les actifs risqués. On tient compte de la possibilité pour l'agent de consommer une partie de sa richesse avant la date T. On note c_t le taux de consommation à la date t, c'est à dire que l'agent consomme $c_t dt$ entre les dates t et $t + dt$. La richesse de l'agent dans ce cadre s'écrit:

$$X_t^{x,c,\pi} = x - \int_0^t c_u e^{-\int_u^t r_s ds} du + \int_0^t e^{-\int_u^t r_s ds} \langle \pi_s, \frac{dS_u}{S_u} \rangle.$$

Les préférences de l'agent et son aversion au risque sont représentées par une fonction d'utilité U de type Von Neumann Morgenstern, supposée croissante, strictement concave, deux fois dérivables et vérifiant les conditions d'Inada:

$$\lim_{x \to 0} U'(x) = \infty \text{ et } \lim_{x \to +\infty} U'(x) = 0.$$

Dans le cas le plus simple, étant donné un horizon $T > 0$, un problème de maximisation d'utilité est formulé de la façon suivante. L'agent investit dans le marché financier et cherche à trouver la stratégie qui maximise l'utilité espérée de sa richesse terminale à la date finale T. Ce problème a été étudié pour la première dois par Merton [Mer69], [Mer71], dans le cas d'un marché complet, et en supposant que l'actif risqué suit une dynamique de Black-Scholes. Il suppose aussi que la fonction d'utilité est une fonction puissance: $U(x) = \frac{x^p}{p}$, pour $x \geq 0$ et $0 \leq p \leq 1$, ou [Sam69] pour un modèle discret. Plus tard, le problème d'investissement optimal dans le cas d'un marché complet avec une fonction d'utilité plus générale a été résolu par Pliska [Pli86].
Dans ce travail, nous attachons une importance particulière à la consommation de l'agent représentatif: nous considérons que l'agent résoud un problème mixte d'investissement et consommation. Plus précisément, il maximise à la fois l'utilité espérée de sa consommation et de sa richesse terminale. Ainsi, si U^1 et U^2 sont deux fonctions d'utilité (U^1 étant de plus supposée dépendre

du temps), le problème de l'agent entre une date 0 et une date T qui est l'horizon du problème s'écrit:

$$\sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}[\int_0^t U^1(t,c_t)dt + U^2(X_T^{x,c,\pi})].$$

Toujours dans le cas d'un marché complet, avec une fonction d'utilité générale, le problème mixte d'investissement/consommation a été traité par Karatzas, Lehoczky et Shreve ou Cox et Huang [CH89]. Pour notre présentation du problème mixte d'investissement/consommation dans ce travail, nous nous inspirerons à plusieurs reprises de la présentation faite par Karatzas et Shreve dans [KS98] ou [KS91].
Ce chapitre revisite le problème d'investissement/consommation en faisant apparaître la place du Growth Optimal Portfolio. Le Growth Optimal Portfolio, ou GOP (parfois appelé numéraire de marché) a été introduit dans un cadre non financier par Kelly [Kel56]. Puis dans un cadre financier, il a été étudié par Long [Lon90], El Karoui, Geman and Rochet [EGR95], Artzner [Art97], Becherer [Bec01], Bajeux-Besnainou et Portait [BBP97]. Une étude détaillée du GOP est faite par Platen et Heath [PH06] et de nombreux articles du même auteur auxquels nous ferons fréquemment référence dans ce deuxième chapitre. Nous verrons que le Growth Optimal Portfolio est un portefeuille particulièrement robuste sur de longues périodes. Par ailleurs, il peut être utilisé comme numéraire pour calculer les prix des zéro-coupons. Il semble donc un outil efficace pour l'étude des taux particulièrement à long terme.
Dans ce chapitre, nous rappelons d'abord la définition du Growth Optimal Portfolio: c'est le portefeuille optimal pour l'utilité logarithmique. Une définition équivalente consiste à trouver le portefeuille pour lequel le drift dans l'équation différentielle stochastique de $\ln S_t^\kappa$ est maximal (dans l'ensemble des portefeuilles $S^\kappa(.)$ strictement positifs). Le Growth Optimal Portfolio est défini à une constante près (sa valeur initiale). Dans ce cas, en notant G_t^* la valeur du Growth Optimal Portfolio à la date t, tel que $G_0^* = 1$, il est solution de l'équation:

$$\frac{dG_t^*}{G_t^*} = r_t dt + \langle \theta_t, dW_t \rangle + ||\theta_t||^2 dt,\ G_0^* = 1,$$

où $\theta(.)$ est le vecteur des primes de risque. Le portefeuille optimal peut s'exprimer comme l'inverse de la densité des prix d'états H_t^0:

$$G_t^* = \frac{1}{H_t^0} = \exp(\int_0^t r_s ds + \int_0^t \langle \theta_t, dW_t \rangle + \frac{1}{2}\int_0^t ||\theta_s||^2 ds).$$

Une propriété cruciale du portefeuille optimal est la suivante. Lorsque que le GOP est utilisé comme numéraire, tous les prix des portefeuilles sont des martingales locales sous la probabilité historique. Cela signifie que tous les portefeuilles actualisés par le GOP sont des martingales locales sous la probabilité historique $\mathbb{P}$.
Dans les sections 2.1 à 2.7 de ce chapitre, nous supposons l'existence d'une probabilité risque-neutre $\mathbb{Q}$, mais nous utilisons le GOP comme numéraire et nous utilisons la probabilité historique comme probabilité de pricing. C'est le même type d'approche que chez Long [Lon90], Bajeux-Besnainou et Protait [BBP97], ou Becherer [Bec01]. Cette approche est très proche de celle de Platen et Heath [PH06] et nous utilisons nombre de ses résultats. La différence principale est que l'approche de Platen ne suppose pas nécessairement l'existence d'une probabilité risque-neutre. Nous revenons sur ce point à la fin du chapitre 2.

Nous revisitons d'abord le problème de maximisation d'utilité espérée de la richesse et de la consommation en faisant apparaître le GOP, en utilisant le fait qu'il s'exprime comme l'inverse de la densité des prix d'états:

$$\sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}[\int_0^t U^1(t,c_t)dt + U^2(X_T^{x,c,\pi})] \text{ s.c. } \mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt + \frac{1}{G_T^*}X_T^{x,c,\pi}\right] \leq x, \tag{0.0.3}$$

où le portefeuille optimal apparaît donc dans la contrainte de budget. La solution de ce problème met en évidence la relation suivante entre le processus de consommation optimale de l'agent représentatif et le GOP:

$$U_c^1(t,c_t^*) = U_c^1(0,c_0^*)\frac{1}{G_t^*}, \tag{0.0.4}$$

où U_c^1 est la dérivée de la fonction d'utilité de la consommation par rapport à sa deuxième variable. Ainsi, on déduit de cette équation des informations sur la forme de la consommation optimale et son lien avec le GOP. En finance, on sait peu de choses sur la forme de la consommation a priori et résoudre le problème de maximisation nous permet de déduire des informations sur la forme de la trajectoire de la consommation optimale. C'est le cheminement inverse en Economie, où des hypothèses sont fréquemment faites sur la forme du processus de consommation (voir par exemple [Gol], où la consommation est supposée avoir une dynamique lognormale).

Puis pour une fonction d'utilité de la forme $U_1(t,c) = e^{-\beta t}u(c)$, et en prenant l'espérance de l'équation précédente (0.0.4) nous obtenons:

$$\exp(-\beta t)\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_t^*)}{u'(c_0^*)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^t r_s ds\right)\right],$$

où nous rappelons que $\mathbb{P}$ est la probabilité historique et $\mathbb{Q}$ est la probabilité risque-neutre. On retrouve ainsi la règle de Ramsey dans le cadre d'un marché financier complet:

Theorem 0.0.1 ***Règle de Ramsey:*** *L'utilité marginale de la consommation optimale et la courbe des taux $R_0(T)$ sont liées par la relation :*

$$R_0(T) = \beta - \frac{1}{T} \log \mathbb{E}^{\mathbb{P}} \left[\frac{u'(c_T^*)}{u'(c_0^*)} \right].$$

Cependant, en marchés complet, la courbe des taux est donnée par le marché, elle est endogène. Dans la règle de Ramsey, le terme à gauche de l'égalité est donné par le marché : la règle de Ramsey indique alors plutôt la façon dont la consommation optimale s'adapte à la fonction d'utilité choisie.
Enfin, en marché complet, la dynamique des zéros-coupons est donnée par l'équation suivante :

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \langle \Gamma(t,T), dW_t + \theta_t dt \rangle, \;\; B(T,T) = 1.$$

où $W(.)$ est un Brownien sous la probabilité historique. Il est possible de redémontrer cette propriété en utilisant le fait que les prix exprimés dans le numéraire du GOP (ou prix benchmarkés) sont des martingales locales.

Jusqu'ici dans ce chapitre nous avons supposé l'existence d'une probabilité risque-neutre, mais nous nous sommes servis du GOP comme numéraire de marché et nous avons utilisé la probabilité historique comme probabilité de pricing. C'est la même approche que dans [Lon90], [BBP97], [Bec01]. Cette approche est assez proche de celle de Platen et utilise un certain nombre de ses résultats de [PH06]. Cependant, la différence principale entre les deux approches est que l'approche de Platen ne nécessite pas l'hypothèse de l'existence d'une probabilité risque-neutre, mais permet tout de même de calculer des prix sous la probabilité historique, en particulier de calculer des prix de zéro-coupons.
Les formules que nous avons données pour la courbe des taux et les prix des zéros-coupons sont vraies pour un horizon T fini. Ainsi nous nous plaçons dans le cas d'un horizon lointain mais fini. Pour le moment dans ce travail nous n'avons pas considéré le cas où $T \to +\infty$. Or en Economie, il n'est pas rare de voir ce cas mentionné dans la littérature. Cependant, en finance, la densité de la probabilité risque-neutre tend vers zéro à l'infini, ce qui a été souligné par Martin [Mar08] et Platen et al. [PH06] (qui utilise alors la probabilité historique sans hypothèses sur la probabilité risque-neutre). A très long terme, le comportement du GOP est dominant, et c'est là qu'il

prend un rôle plus important. Dans le cadre de cette thèse, nous considérons cependant un horizon long mais fini. Le cas asymptotique reste une question à explorer dans des travaux futurs.

Dans les paragraphes de la fin de ce chapitre introductif, nous présentons les résultats obtenus par Platen et Heath concernant les questions d'approximations du GOP et de pricing sous la probabilité historique dans ce cadre, dans [PH06] et d'autres travaux du même auteur, en particulier: [Pla06, Pla04a, Pla04b, Pla09, PR09, Pla04c].
Si on croit à un modèle ou à une approximation pour le GOP entre les dates t et T (noté $G^*_{t,T}$), Platen en déduit des prix de zéro-coupons:

$$B(t,T) = \mathbb{E}\left[\frac{1}{G^*_{t,T}}|\mathcal{F}_t\right].$$

Nous en déduisons le résultat suivant. Si on croit à un modèle ou à une approximation pour le GOP between dates t and T alors il est possible d'en déduire une courbe des taux "historique" $R_t^{GOP}(\theta)$ déduite de données observables:

$$R_t^{GOP}(T) = -\frac{1}{T-t}\ln\mathbb{E}^{\mathbb{P}}\left[\frac{1}{G^*_{t,t+T}}|\mathcal{F}_t\right]$$

Cette formule est donc, après la Ramsey Rule, une autre expression de la courbe des taux. La différence principale est cependant qu'il n'est pas nécessaire de faire d'hypothèses sur la forme des fonctions d'utilité. Une perpective intéressante serait de tester des approximations du GOP avec des données réelles et examiner et comparer les courbes de taux qui en résultent.

Maximisation d'utilité et taux d'intérêt long terme en marché incomplet

Par la suite, de très nombreuses études ont entrepris de s'affranchir des limites de la formulation de Merton, en particulier de l'hypothèse d'un marché complet qui n'est pas réaliste et qui est trop restrictive pour de nombreuses applications. En quelques mots ici, nous faisons référence aux travaux existants concernant le marché incomplet.
L'incomplétude peut provenir des contraintes sur les portefeuilles admissibles, voir Cvitanic et Karatzas [CK92] ou Zariphopoulou [Zar94]. Par exemple dans [KS98, KLSX91, HK04], ou [Mra09], ces contraintes font que le portefeuille de l'agent doit se trouver dans un cône convexe K. Le pricing en marché incomplet a aussi été abordé par [EQ95]. D'un point de vue plus général, le marché incomplet a été étudié par He et Pearson [HP91] puis Karatzas, Lehoczky, Shreve et Xu [KLSX91]. Les travaux plus récents de Kramkov et Schachermayer concernent un cadre encore plus général où les prix des actifs sont seulement supposés être des semi-martingales [KS99, KS03].
Dans la présentation du marché incomplet que nous avons choisi de prendre dans cette thèse, l'incomplétude vient des contraintes sur le portefeuille de l'agent. Nous avons choisi des contraintes sur le portefeuille qui sont simples: l'agent représentatif peut investir dans certains actifs et pas dans les autres: il peut investir dans l'actif sans risque et dans M actifs risqués parmi N sources de bruit dans le marché, avec $M < N$. Notre point de vue est très proche de celui décrit dans le Chapitre 6 de [KS98].
Plus précisément, dans ce marché financier, il y a un actif sans risque de prix $S^0(.)$ donné $dS_t^0 = S_t^0 r_t dt$ (où $r(.) \geq 0$ est le taux court) et M actifs risqués dans lesquels l'agent peut investir. La dynamique de leurs prix $S^i(.), i = 1, \dots, M$ est donnée par:

$$\frac{dS_t^i}{S_t^i} = \tilde{b}_t^i dt + \langle \tilde{\sigma}_t^i, dW_t \rangle,$$

où le drift $\tilde{b}(.)$ est un vecteur de taille M, $\tilde{\sigma}(.)$ est une matrice de volatilité de taille $N \times M$, où $\tilde{\sigma}^i(.)$ est le i-ème vecteur ligne. Nous supposons que $\sigma\sigma^{\mathbf{T}}(t, \omega)$ est inversible (et $^{\mathbf{T}}$ dénote la transposée d'une matrice). Nous appelons $\tilde{\theta}(.)$ le vecteur des primes de risques minimal:

$$\tilde{\theta}_t = \tilde{\sigma}_t^{\mathbf{T}} (\tilde{\sigma}_t \tilde{\sigma}_t^{\mathbf{T}})^{-1} (\tilde{b}_t - r_t \mathbf{1}_M),$$

Nous appelons $\pi_t^i, i = 1, \dots, M$ la fraction de richesse de l'agent investie dans chacun des actifs risqués. Le vecteur $\kappa_t := \sigma_t^{\mathbf{T}} \pi_t$ a une importance particulière. Soit un agent représentatif qui part d'une richesse initiale x, investit dans un portefeuille π et consomme une partie de sa richesse au taux c. Dans la suite nous considérons des processus de richesse positive dont la valeur à la date t est donnée par $X^x x, c, \kappa$. Pour un problème avec consommation l'équation d'auto-financement s'écrit, en fonction de κ_t de la manière suivante:

$$dX_t^{x,c\kappa} = -c_t dt + X_t^{x,c,\kappa} r_t dt + X_t^{x,c,\kappa} \langle \kappa_t, dW_t + \tilde{\theta}_t dt \rangle, \; X_0^{x,c,\kappa} = x.$$

Nous appelons $\mathcal{K}_t$, l'image de $\sigma_t^{\mathbf{T}}$. En particulier, pour tout $0 \leq t \leq T$, le vecteur des primes de risque minimal $\tilde{\theta}_t$ et le vecteur κ_t sont dans $\in \mathcal{K}_t$. L'orthogonal de cet espace joue également un rôle fondamental. Nous notons $\mathcal{K}_t^{\perp}$ l'orthogonal de $\mathcal{K}_t$ dans $\mathbb{R}^n$. Pour $\nu(.) \in \mathcal{K}^{\perp}$, on appelle densité des prix d'états un processus $H^{\nu}(.)$ tel que le processus $H_t^{\nu} X_t^{x,c,\kappa} + \int_0^t H_s^{\nu} c_s ds$ est une $\mathbb{P}$ martingale locale. Le processus de densité des prix d'états est alors un processus vérifiant:

$$dH_t^{\nu} = -r_t dt - \langle \tilde{\theta}_t + \nu_t, dW_t \rangle, \text{ and } H_0^{\nu} = 1, \; \nu(.) \in \mathcal{K}^{\perp}.$$

En particulier nous rappelons que le vecteur des primes de risques minimal $\tilde{\theta}(.)$ et le processus dual $\nu(.)$ sont orthogonaux.
Le processus $(Y_t^{\nu}(y))_{t \geq 0}$ sera également appelé densité des prix d'états, mais il fait de plus apparaître la condition initiale $y > 0$:

$$Y_t^{\nu}(y) = y H_t^{\nu} = y \exp(-\int_0^t r_s ds - \int_0^t \langle \tilde{\theta}_s + \nu_s, dW_s \rangle - \frac{1}{2} \int_0^t ||\tilde{\theta}_s + \nu_s||^2 du),$$

et $Y_0^{\nu}(y) = y$. Pour résoudre un problème de maximisation d'utilité en marché incomplet lorsque l'incomplétude provient de contraintes sur les portefeuilles, nous adoptons la démarche de [KS98]. Il s'agit de considérer une famille de marchés auxiliaires $\mathcal{M}_{\nu}$, pour chaque processus dual $\nu(.) \in \mathcal{K}^{\perp}$, dans lesquels il n'y a plus de contraintes et qui sont construits en complétant le marché $\tilde{\mathcal{M}}$ par des actifs fictifs.
Nous présentons ensuite la manière dont la maximisation d'utilité peut être abordée par dualité en marché incomplet, avec une présentation proche de celle de [Pha07]. Nous n'entrerons pas dans les détails, nous présenterons seulement les résultats principaux existants sur le sujet. Cependant, il est important de les rappeler, car c'est le problème dual que nous allons considérer par la suite. Pour un problème mixte d'investissement/consommation, le problème dual s'écrit, pour tout $y > 0$:

$$\tilde{V}(y) = \inf_{\nu \in \mathcal{K}^{\perp}} \mathbb{E}[\int_0^T \tilde{U}^1(t, Y_t^{\nu}(y)) dt + \tilde{U}^2(Y_T^{\nu}(y))].$$

L'optimum de ce problème est le processus dual optimal ν^*. Il est alors possible de définir une mesure de probabilité $\mathbb{Q}^{\nu^*}$ telle que le processus $(W_t^{\mathbb{Q}^{\nu^*}} := W_t + \int_0^t \tilde{\theta}_s + \nu_s^* ds)_{t\geq 0}$, est un $(\mathbb{F}, \mathbb{Q}^{\nu^*})$ mouvement Brownien standart.
En marché incomplet, la probabilité de pricing optimale associée $\mathbb{Q}^{\nu^*}$ n'est plus universelle. Elle peut dépendre de la maturité, de la fonction d'utilité choisie (par exemple à travers son paramètre de préférence pour le présent β) et de la richesse dans l'économie y: nous la notons $\nu^*(y)$ dans la suite.
Par ailleurs, dans le chapitre précédent, nous avons décrit les propriétés du GOP ou numéraire de marché dans le cadre d'un marché complet. Nous souhaitons relier le GOP au cas du marché incomplet puis aux résultats des Chapitres suivants. Nous définissons un GOP en marché incomplet (spécifique du marché incomplet tel que nous l'avons défini : M actifs dans lesquels il est possible d'investir parmi N). Notre définition est analogue au cas du marché complet. Le GOP en marché incomplet est le portefeuille optimal pour l'utilité logarithmique:

$$\tilde{G}_t^\theta = \exp\left(\int_0^t r_s ds + \int_0^t \langle\tilde{\theta}_s, dW_s\rangle + \frac{1}{2}\int_0^t ||\tilde{\theta}_s||^2 dt\right).$$

Il est possible de construire ce portefeuille optimal à partir des actifs tradables uniquement, en constituant un portefeuille composé des M actifs risqués tradables, les fractions étant données par $\kappa_t = \tilde{\theta}_t$ et le reste en actif non risqué.
Nous examinons enfin les conséquences de l'incomplétude du marché sur la structure des taux. En marché incomplet, la question du pricing des zéro-coupons se pose. Nous utilisons alors une règle de pricing liée à l'utilité marginale : par des prix de Davis. Ainsi, il est possible de définir des prix de zéro-coupons en marché incomplet. Nous noterons $B^{\nu^*(y)}(t,T)$ pour le prix à la date t pour le zéro-coupon de maturité T.

$$B^{\nu^*(y)}(t,T) = \mathbb{E}[H_{t,T}^{\nu^*(y)}] = \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t]. \tag{0.0.5}$$

Nous notons par $R_0^{\nu^*(y)}(T)$ la courbe des taux correspondante, définie de façon à ce que pour tout T, $B^{\nu^*(y)}(0,T) = e^{-TR_0^{\nu^*(y)}(T)}$.

Proposition 0.0.1 *En marché incomplet, en adoptant une règle de pricing par des prix de Davis, il est possible de donner le lien entre la courbe des taux et l'utilité marginale de la consommation optimale (ici en choisissant $U^1(t,c) = exp(-\beta t)u(c)$). C'est la règle de Ramsey en marché incomplet.*

$$R_0^{\nu^*}(T) = \beta - \frac{1}{T}\log\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^{\nu^*})}{u'(c_0^{\nu^*})}\right]. \tag{0.0.6}$$

Il est donc montré que la règle de Ramsey reste valable si on adapte une règle de pricing par des prix de Davis. Ainsi, nous définissons une courbe des taux en marché incomplet correspondant à des prix de Davis, elle n'est valable que pour des petits nominaux. Par ailleurs, comme le processus dual optimal ν^* dépend de y la richesse dans l'économie, la courbe des taux aussi. Nous avons commencé notre étude du long terme avec la règle de Ramsey, elle sert de fil conducteur à ce travail. Donc nous avons cherché à l'établir dans le cas du marché complet puis incomplet. Cependant elle s'établit en utilisant un prix de Davis qui est un prix marginal, valable seulement pour des petits nominaux. Dans les autres cas, une perpective serait d'employer un pricing par indifférence.
Enfin, nous examinons la différence entre les prix de zéro-coupons $B^{\nu^*}(t,T)$ et les prix de zéro-coupons calculés à partir des actifs tradables uniquement et notés $B^{GOP}(t,T) = \mathbb{E}[e^{-\int_t^T r_s ds}\mathcal{E}(-\int_0^t \langle\tilde{\theta}_s, dW_s\rangle)|\mathcal{F}_t]$. Nous examinons de quelle façon la courbe des taux est modifiée en marché incomplet.

Proposition 0.0.2 *En marché incomplet, la dynamique des zéros-coupons $B^{\nu^*}(t,T)$ s'écrit:*

$$B^{\nu^*(y)}(t,T) = B^{GOP}(t,T)\mathbb{E}^{\mathbb{Q}^{GOP}}\left[\mathcal{E}\left(-\int_t^T \langle \nu_s^*(y), dW_s\rangle\right)|\mathcal{F}_t\right],$$

où nous définissons une probabilité $\mathbb{Q}_T^{GOP}$ qui joue un rôle similaire à une probabilité forward neutre, et dont la densité par rapport à la probabilité historique $\mathbb{P}$ est définie par:

$$\frac{d\mathbb{Q}_T^{GOP}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T\langle\tilde{\theta}_s, dW_s\rangle\right)}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T\langle\tilde{\theta}_s, dW_s\rangle\right)\right]}.$$

Deux généralisations: horizon aléatoire et incertitude sur un paramètre du modèle

Les principaux résultats de cette partie sont dans les chapitres 4 et 5.
Différentes généralisations sont proposées dans le Chapitre 4. Ces deux généralisations font intervenir des grossissements de filtrations. Plus précisément dans le premier cas il s'agit d'un grossissement de filtration progressif, qui vient de l'introduction d'un horizon aléatoire ζ. Dans le deuxième cas il s'agit d'un grossissement de filtration initial, c'est pourquoi nous les présentons ensemble dans ce chapitre.

Le $\mathbb{G}$-marché: un nouveau point de vue sur la consommation

Dans ce chapitre, nous commençons d'abord par présenter de nouveaux résultats sur la manière d'interpréter la consommation. Nous expliquons comment il est possible de voir la consommation comme des provisions à utiliser en cas de défaut.
Tout d'abord rappelons que dans le cas du problème classique de maximisation d'utilité de la consommation et de la richesse terminale, l'expression à maximiser est:

$$\sup_{(c,\kappa)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T U^1(t,c_t)dt + U^2(T, X_T^{x,c,\kappa})\right].$$

Dans cette equation, nous voyons le statut différent des deux fonctions d'utilité. La fonction U^1 est fonction d'un **taux de consommation** alors que la fonction d'utilité de la richesse terminale U^2 est fonction d'une richesse aggrégée. Or souvent ce sont les mêmes fonctions d'utilité qui sont choisies pour ces deux quantités (par exemple des fonctions puissance). Dans ce chapitre, nous nous demandons comment il est possible de donner un même statut à ces quantités, c'est-à-dire si on peut voir la consommation comme une certaine quantité de richesse.
Nous réinterprétons la consommation de la manière suivante. Le taux de consommation c_t est maintenant considéré comme une accumulation de réserves faites par l'agent représentatif au cours du temps. Ainsi, au lieu d'investir et de consommer, l'agent représentatif investit et met de côté une partie de sa richesse. Ainsi c_t représente une certaine quantité de richesse, une certaine quantité de provisions. Ces provisions sont mise de côté pour faire face à un évènement imprévisible. Si cet évènement ne se produit pas avant la date de maturité T, l'agent maximise sa richesse terminale (comme pour un problème

classique de gestion de portefeuille et de maximisation de la richesse terminale). Si l'évènement se produit avant T, il perd son portefeuille mais garde ses provisions en cash. Il maximise alors l'utilité de cette richesse.
Pour modéliser cet évènement imprévisible, il faut introduire une nouvelle quantité qui n'est pas dans le marché, qu'on modélise par une variable aléatoire ζ. Par exemple, dans un cas très simple, une variable exponentielle, sans mémoire, pourrait être utilisée. Nous appellons aussi cette date aléatoire ζ temps de défaut, comme dans le cadre du risque de crédit.
Plus précisément, l'information sur les prix des actifs est contenue dans la filtration $\mathbb{F}$. A la date aléatoire ζ, les provisions sont utilisées. La variable aléatoire ζ is contenue dans une nouvelle filtration $\mathbb{G}$. Dans ce chapitre nous introduisons et nous décrivons d'abord la filtration $\mathbb{G}$. Soit $0 < \zeta < \infty$ une variable aléatoire positive. Soit $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ la plus petite filtration pour laquelle τ est un temps d'arrêt, i.e. $\mathcal{D}_t = \mathcal{D}^0_{t+}$ avec $\mathcal{D}^0_t = \sigma(\zeta \wedge t)$. Puis nous considérons la filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ telle que:

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t,$$

et $\mathcal{G}_0$ contient les négligeables de $\mathcal{G}_\infty$. Cette filtration est souvent utilisée dans la littérature concernant le risque de crédit pour modéliser l'information globale dans le marché (contenue dans $\mathbb{F}$ and ζ), par exemple voir [Jea, EJY00].

Hypothesis 0.0.1 *Dans ce travail, nous sommes dans le cadre de l'hypothèse (H).*

Nous définissons le processus de survie:

$$\mathbb{P}[\zeta > t|\mathcal{F}_\infty] = e^{-\int_0^t \varphi_s ds} = e^{-\Phi_t},$$

où $\Phi(.)$ est un processus croissant, $\mathbb{F}$-adapté, tel que $\Phi_\infty = \infty$ et ici supposé dérivable.
Dans ce chapitre nous décrivons un $\mathbb{G}$-marché dans lequel l'agent représentatif maximise l'utilité de sa richesse terminale ou de ses provisions. Nous décrivons d'abord un processus de richesse dans le $\mathbb{G}$-marché. Nous considérons un agent représentatif qui part d'une richesse initiale x, de portefeuille $\kappa(.)$ et de taux de consommation $c(.)$. En partant de la richesse associée dans le $\mathbb{F}$-marché, notée $X^{x,c,\kappa}(.)$ (un processus $\mathbb{F}$-progressivement mesurable) nous construisons un processus de richesse admissible dans le $\mathbb{G}$-marché (cette fois un processus $\mathbb{G}$-progressivement mesurable). C'est une manière de donner une intuition sur la construction du $\mathbb{G}$-marché.

Definition 0.0.1 ***Richesse dans le $\mathbb{G}$-marché:*** *Nous considérons un agent représentatif de richesse initiale x, et $X^{x,c,\kappa}(.)$ la richesse associée dans le $\mathbb{F}$-marché. We définissons le processus $\mathbb{G}$-progressivement mesurable $X^{\mathbb{G},x,\kappa}(.)$ comme:*

$$X_t^{\mathbb{G},x,\kappa} = X_t^{x,c,\kappa} e^{\Phi_t} \textit{ for } t < \zeta$$
$$X_\zeta^{\mathbb{G},x,\kappa} = c_\zeta e^{\Phi_\zeta} (\varphi_\zeta)^{-1}.$$

Ce nouveau processus de richesse $X^{\mathbb{G},x,\kappa}(.)$ a un saut à la date ζ. Les termes en e^{Φ_t} disparaissent lorsqu'on prend les espérances de ces processus.

Nous pouvons vérifier que pour des richesses $\check{X}^{x,c,\kappa}(.)$ admissibles, les richesses définies dans le $\mathbb{G}$ sont positives presque surement:

$$X_t^{\mathbb{G},x,\kappa} \geq 0, \mathbb{P} - a.s.$$

Avec cette formulation, nous interprétons le taux de consommation comme une certaine quantité de cash. Pour $0 \leq t < \zeta$, la gestion du portfolio se fait de façon classique, mais à chaque date t, une certaine quantité de richesse est mise en réserve et constitue les provisions. Si l'évènement se produit à une date ζ avant la date T, l'investisseur utilise ses provisions. A l'instant du défaut, toute la partie portefeuille est liquidée et ce qui reste est remis en cash. Le saut en ζ est négatif: $X_{\zeta^-}^{\mathbb{G},x,\kappa} \geq X_\zeta^{\mathbb{G},x,\kappa}$ p.s. L'agent perd une partie de sa richesse à l'instant de défaut, mais il lui reste ses provisions à la place (qui sont exprimées en unités de l'actif sans risque). La dynamique de la richesse dans le $\mathbb{G}$-marché s'écrit alors:

$$dX_t^{\mathbb{G},x,\kappa} = X_t^{x,c,\kappa}(r_t dt + \langle \kappa_t, dW_t + \tilde{\theta}_t \rangle) e^{\Phi_t} \mathbf{1}_{t<\zeta} - (X_{t^-}^{\mathbb{G},x,\kappa} - X_t^{\mathbb{G},x,\kappa}) \frac{dL_t^{\mathbb{G}}}{L_{t^-}^{\mathbb{G}}},$$

où $L_t^{\mathbb{G}} = e^{\Phi_t} \mathbf{1}_{t<\zeta}$. Le premier terme dans la dynamique de la richesse tient compte du fait que l'agent investit une partie de sa richesse dans les actifs de base du marché. Nous supposons en outre que l'agent peut investir dans un actif de prix $L_t^{\mathbb{G}}$ (de type Credit Default Swap). Avec cette hypothèse supplémentaire, il n'y a plus d'incomplétude venant de la variable aléatoire ζ, l'incomplétude éventuelle vient uniquement de contraintes sur les portefeuilles et $X_t^{\mathbb{G},x,\kappa}$ est une stratégie d'investissement admissible dans le $\mathbb{G}$-marché.
Nous définissons ensuite une fonction d'utilité dans le $\mathbb{G}$-marché.

Definition 0.0.2 ***Fonction d'utilité dans le $\mathbb{G}$-marché:*** *Nous considérons un agent représentatif avec une structure de préférence (U^1, U^2)*

et $0 < \zeta < \infty$ une variable aléatoire. Nous définissons $U^{\mathbb{G}}(t,x)$ such that for all $x > 0$:

$$U^{\mathbb{G}}(t,x) = U^2(t, xe^{-\Phi_t})e^{\Phi_t}\mathbf{1}_{t<\zeta} + U^1(t, ce^{-\Phi_t})e^{\Phi_t}(\varphi_t)^{-1}\mathbf{1}_{\zeta \le t} \ a.s.$$

Cette fonction est croissante et concave, il s'agit bien d'une fonction d'utilité. Sa particularité est d'avoir un saut en ζ: il s'agir d'un exemple de fonction d'utilité stochastique. Ainsi, dans le $\mathbb{G}$-marché, il y a un saut dans le portefeuille $X^{\mathbb{G}}$ et dans l'utilité $U^{\mathbb{G}}$. Avec cette définition, la variable aléatoire ζ introduit un horizon aléatoire.

Ici également, des termes de la forme $e^{-\Phi_t}$ disparaissent lorsqu'on intègre la fonction d'utilité. D'autre part la différence entre les deux termes de l'équation précédente vient des différents statuts de U^1 and U^2 (on considère l'intégrale de l'utilité de la consommation U^1 mais seulement la valeur terminale de U^2).

Nous appelons $\mathcal{A}^{\mathcal{G}}(x)$ l'ensemble des stratégies admissibles dans le $\mathbb{G}$-marché partant d'une richesse initiale x.

Theorem 0.0.2 *Le problème de maximisation d'utilité dans le $\mathbb{G}$-marché s'écrit:*

$$\sup_{\kappa \in \mathcal{A}^{\mathcal{G}}(x)} \mathbb{E}[U^{\mathbb{G}}(T, X_T^{\mathbb{G},x,\kappa})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta \le T}].$$

Il y a équivalence entre ce problème de maximisation d'utilité dans le $\mathbb{G}$-marché et le problème d'investissement/consommation dans le $\mathbb{F}$-marché. Mais ici on s'est ramené à un problème de maximisation de la richesse terminale.

Nous passons ensuite à l'expression du problème dual dans le $\mathbb{G}$-marché. Une fonction d'utilité duale dans le $\mathbb{G}$-marché est définie par:

$$\tilde{U}^{\mathbb{G}}(t,y) = \inf_{x>0}\{U^{\mathbb{G}}(t,x) - xy\},$$

c'est-à-dire la transformée de Fenchel de la fonction $U^{\mathbb{G}}$. Il s'agit bien d'une fonction d'utilité duale.

Theorem 0.0.3 *Le problème dual dans le $\mathbb{G}$-marché est donné par:*

$$\sup_{\nu \in \mathcal{K}^\perp} \mathbb{E}[U^{\mathbb{G}}(T, Y_T^\nu(y))\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, Y_\zeta^\nu(y))\mathbf{1}_{\zeta \le T}].$$

Le problème dual dans le $\mathbb{G}$-marché peut également se ramener au problème dual d'investissement/consommation dans le $\mathbb{F}$-marché.

Incertitude sur un paramètre du modèle

La deuxième partie de ce chapitre concerne le cas où il y a une incertitude sur un paramètre du modèle. Cette hypothèse est particulièrement pertinente pour un univers d'investissement où l'horizon est très long, et elle apparaît dans la littérature économique sur le sujet (voir par exemple Gollier [Gol09c]). L'agent représentatif a par exemple une incertitude sur la croissance de l'économie pour les années suivantes.
Pour modéliser simplement l'ambiguité sur un paramètre, on utilise une variable aléatoire L (de distibution μ^L).

L'agent connaît la loi de L Dans un premier temps, L est indépendante de la filtration $\mathbb{F}$. On suppose pour commencer que l'agent représentatif ne connaît pas la réalisation de L, mais connaît sa distribution μ^L et va s'en servir dans ses choix. La fonction d'utilité de l'agent $U(t,c)$ s'écrit alors comme la sup-convolution des fonctions $U^l(t,c)$ correspondant chacune au cas où la réalisation $L = l$. La même relation de sup-convolution peut être montrée pour les fonctions valeurs correspondantes. Plus précisément les fonctions valeurs duales sont liées par:

$$\tilde{V}(y) = \int_{\mathbb{R}} \tilde{V}^l(y) d\mu^L(l).$$

L'agent connaît L Une autre situation, que nous examinons ici, est celle où l'agent connaît L après la date 0. Cette fois, la variable aléatoire L n'est plus supposée indépendante de $\mathbb{F}$. Ce cas fait appel à des références concernant le grossissement de filtration initial [Jac79, Jac85, GP98] ou alors [Hil04, HJ10]. Il est alors possible d'utiliser des résultats qui donnent la trajectoire de la consommation optimale $c_t^{L,*} = I_1(t, \mathcal{Y}^L(x_0^L) H_t^L)$, où $\mathcal{Y}^L(x_0^L)$ est le multiplicateur de Lagrange ($\sigma(L)$-mesurable ici). Nous examinons ensuite les conséquences sur la courbe des taux pour l'agent qui a de l'information sur L. Ici pour simplifier on suppose le $\mathbb{F}$-marché origine complet.

Proposition 0.0.3 *L'expression de la dynamique des zéro-coupons pour l'agent qui connaît L, de prix $B^L(t,T)$, et sa comparaison avec les zéro-coupons (de prix $B(t,T)$) fait apparaître la probabilité forward-neutre $\mathbb{Q}_T$ de la manière suivante:*

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}}|_{\mathcal{F}_T} = \frac{exp(-\int_0^T r_s ds)}{B(0,T)}.$$

$$B^L(t,T) = B(t,T)\mathbb{E}^{Q_T}[exp(-\int_t^T \langle \rho_s^L, dW_s\rangle + \frac{1}{2}\int_t^T ||\rho_s^L||^2 ds)|\mathcal{F}_t],$$

où $\rho^L(.)$ *est le drift d'information (dépend de L).*

Une perspective pour cette partie est de reformuler ces questions dans le cadre plus général du problème d'investissement/consommation en marché incomplet, puis d'intégrer le cas où les agents ont des croyances différentes suivant la valeur de L. Les densités de prix d'état sont alors différentes pour chaque réalisaition $L = l$.

Utilités dynamiques et taux d'intérêt

Le but de ce chapitre est d'adapter les résultats précédents concernant la dynamique de la courbe des taux et le $\mathbb{G}$-marché au cas des fonctions d'utilité dynamiques.
Dans l'approche standart de la maximisation d'utilité espérée, l'agent représentatif se fixe un horizon de gestion, une fonction d'utilité concave et croissante qui traduit son aversion au risque, et dont il maximise l'espérance. L'agent représentatif choisit sa stratégie optimale et n'en change pas, le critère de préférence ne change pas non plus. C'est ce que nous avons fait jusque là dans cette thèse. Cette approche a ses limites pour le long terme. En effet, sur une très longue période, lorsque les paramètres de l'économie changent de façon importante, on ne peut pas ignorer que l'aversion au risque de l'agent représentatif évolue aussi. Par ailleurs, plus la période de temps qu'on étudie est longue, plus il est probable que les préférences de l'investisseur changent durant ce temps. Pour notre étude il est donc crucial de tenir compte du fait que les préférences de l'agent représentatif et donc sa fonction d'utilité changent au cours du temps. Il n'est cependant pas évident de savoir a priori quelle peut-être la forme de ces fonctions d'utilité. Cela motive l'uilisation de fonctions d'utilité dynamiques.

Plusieurs situations dans la littérature montrent des fonctions d'utilité qui changent au cours du temps. C'est le cas des fonctions d'utilités récursives de la forme $Y_t^c = \mathbb{E}[\int_t^T f(c_s, Y_s^c)ds|\mathcal{F}_t]$ étudiées par [DE92b, DE92a], et [LQ03, LZ04].
Au cours des chapitres précédents, nous avons également vu que des fonctions d'utilité stochastiques apparaissent naturellement, comme la fonction d'utilité dans le $\mathbb{G}$-marché, $U^{\mathbb{G}}(t,x) = U^2(t,x)\mathbf{1}_{t<\zeta} + U^1(t,\zeta)\mathbf{1}_{\zeta\leq t}$. Cet exemple suggère des utilités stochastiques.
Enfin des travaux récente de Musiela et Zariphopoulou ou El Karoui et Mrad ([Mra09] par exemple) ou Berrier et al. [BT08] concernent les fonctions d'utilité dynamiques. Dans le dernier chapitre de cette partie, nous examinons donc la courbe des taux dans le cadre de ces utilités dynamiques.

Utilités dynamiques : un état de l'art En 2002, Musiela et Zariphopoulou ont proposé un point de vue nouveau sur les fonctions d'utilité, ils ont introduit la notion de "forward utility". Il s'agit d'un champ aléatoire $u(t,x)$ adapté à l'information disponible, qui est à chaque instant un utilité standart. Il s'agit d'une utilité dynamique, progressive, cohérente avec un marché financier donné. Ces fonctions d'utilité sont indépendantes de l'horizon d'investissement. Par la suite, elles ont également

été étudiées par Berrier, Rogers et Tehranchi [BRT07], et El Karoui et Mrad [Mra09, EM10, KM10b, KM10a].
Dans ce chapitre nous rappelons la définition des utilités progressives consistantes dynamiques (que nous appellerons parfois seulement utilités dynamique par raccourci), telle qu'elle est donnée par [Mra09], page 135. Il s'agit d'une utilité dynamique de la richesse. Nous nous appuierons sur un certain nombre de résultats de [Mra09, EM10, KM10b, KM10a]. Par la suite nous souhaitons aussi utiliser des utilités dynamiques de la consommation.

Résultats obtenus Au cours de ce travail, nous avons souligné plusieurs fois l'importance du processus de consommation dans notre étude. Aussi, la dernière contribution de cette partie se propose de généraliser au cas des utilités avec consommation les nouvelles méthodes introduites par Musiela et Zariphopoulou pour les richesses grâce au concept d'utilités progessives consistantes.

Definition 0.0.3 ***Utilités dynamiques de la richesse et de la consommation:***
Soient deux champs aléatoires U^1 et U^2 sur $[0,+\infty[\times\mathbb{R}\times\Omega \rightarrow \mathbb{R}^+$ tels que pour tout $t \geq 0$, $U^1(t,.)$ et $U^2(t,.)$ sont croissantes et concaves et $\mathbb{E}[U^2(t,x)] < +\infty$ and $\mathbb{E}[U^1(t,c)] < +\infty$. A la date 0, $U^2(0,x) = u^2(x)$ et $U^1(0,c) = u^1(0,c)$ sont des fonctions d'utilité standart.
Pour toutes les richesses admissibles $X^{x,c,\kappa}(.)$ et les processus de consommation associés (on note $(X^{x,c,\kappa}(.), c(.)) \in \mathfrak{X}$), on a la propriété suivante:

$$U^2(t, X_t^{x,c,\kappa}) + \int_0^t U^1(s, c_s)ds, \tag{0.0.7}$$

est une surmartingale. Et il existe un optimum $(X^*(.), c^*(.))$ pour lequel ce processus est une martingale. Alors (U^1, U^2) sont des fonctions d'utilité dynamiques $\mathfrak{X}$-consistentes. Nous les appellerons aussi structure de préférence dynamique ou paire d'utilités dynamiques.

Des fonctions d'uilité dynamiques de la consommation ont été étudiées Berrier and Tehranchi [BT08], notre définition est légèrement différente, en particulier elle suppose que ces fonctions d'utilité sont positives. Nous définissons ensuite des utilités duales dynamiques.

Proposition 0.0.4 *Les champs aléatoires $(\tilde{U}^1, \tilde{U}^2)$, transformées de Fenchel des fonctions d'utilités dynamiques (U^1, U^2) vérifient les propriétés suivantes.*

- *Pour tout $t > 0$, $\tilde{U}^1(t,.)$ and $\tilde{U}^2(t,.)$ sont convexes décroissantes*

- *Pour tout processus de densité de prix d'état $Y^\nu(y)$, le processus suivant est une sous-martingale:*

$$\tilde{U}^2(t, Y_t^\nu(y)) + \int_0^t \tilde{U}^1(s, Y_s^\nu(y))ds. \tag{0.0.8}$$

- *Et il existe un optimum ν^*, tet que $\tilde{U}^2(t, Y_t^*(y)) + \int_0^t \tilde{U}^1(s, Y_s^*(y))ds$ est une martingale.*

Alors $(\tilde{U}^1, \tilde{U}^2)$ est une paire de ***fonctions d'utilité duales dynamiques****.*

L'avantage de la formulation duale est qu'il y a qu'un seul processus test, la densité des prix d'états, au lieu du couple richesse/consommation.
Ensuite, comme dans le Chapitre 4, nous étendons la situation du $\mathbb{G}$-marché au cas des utilités dynamiques.

Theorem 0.0.4 *Soit un agent représentatif avec une stucture de préférence dynamique (U^1, U^2). Soit $0 < \zeta < \infty$ une variable aléatoire. Nous définissons la fonction d'utilité $U^{\mathbb{G}}(t, x)$:*

$$U^{\mathbb{G}}(t, x) = U^2(t, xe^{-\Phi_t})e^{\Phi_t}\mathbf{1}_{t<\zeta} + U^1(\zeta, ce^{-\Phi_\zeta})e^{\Phi_\zeta}(\varphi_\zeta)^{-1}\mathbf{1}_{\zeta\leq t} \ a.s.$$

Pour toutes les richesses admissibles $X^{\mathbb{G},x,\kappa}(.)$ dans le $\mathbb{G}$-marché, on note$X^{\mathbb{G},x,\kappa}(.) \in \mathfrak{X}^{\mathbb{G}}$. Alors $U^{\mathbb{G}}(t, x)$ est une ***utilité dynamique dans le $\mathbb{G}$-marché*** *$\mathfrak{X}^{\mathbb{G}}$-consistante. C'est à dire, $U^{\mathbb{G}}$ est un champ aléatoire, pour tout $t > 0$, $U^{\mathbb{G}}(t, .)$ est croissante et concave, $\mathbb{E}[U^{\mathbb{G}}(t, x)] < +\infty$. Pour toutes les richesses test $X^{\mathbb{G},x,\kappa}(.) \in \mathfrak{X}^{\mathbb{G}}$, le processus:*

$$t \to U^{\mathbb{G}}(t, X_t^{\mathbb{G},x,\kappa})\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta\leq t}, \tag{0.0.9}$$

est une surmartingale et il existe an optimum pour lequel c'est une martingale.

Une fois de plus, la formulation duale est plus simple. Les processus test sont les densités des prix d'état $Y^\nu(y)$ qui sont les mêmes que dans le $\mathbb{F}$-marché. C'est donc la formulation duale qui sera privilégiée.

Proposition 0.0.5 *La fonction $\tilde{U}^{\mathbb{G}}(t, .)$ est une fonction d'utilité duale dynamique dans le $\mathbb{G}$-marché, alors $t \to \tilde{U}^{\mathbb{G}}(t \wedge \zeta, Y_{t\wedge\zeta}^\nu(y))$ est une sous-martingale:*

$$\mathbb{E}[\tilde{U}^{\mathbb{G}}(T \wedge \zeta, Y_{T\wedge\zeta}^\nu(y))|\mathcal{G}_t] \geq \tilde{U}^{\mathbb{G}}(t \wedge \zeta, Y_{t\wedge\zeta}^\nu(y)).$$

Et il existe un optimum ν^*, pour lequel le processus $t \to \tilde{U}^{\mathbb{G}}(t \wedge \zeta, Y_{t\wedge\zeta}^*(y))$ est une $\mathbb{G}$-martingale.

Etant donné un processus dual optimal, une structure de préférence standart, une richesse optimale dans le $\mathbb{G}$-marché $X^{\mathbb{G}}$ il alors est possible de construire explicitement les utilités dynamiques dans le $\mathbb{G}$-marché associées (d'une façon similaire à ce qui est fait dans [KM10b], pour les fonction d'utilité dynamiques de la richesse).
Puis, en utilisant les résultats des Chapitres précédents, la dernière étape consiste à nous intéresser à la courbe des taux. Dans ce chapitre, la densité des prix d'états est solution de l'équation suivante:

$$\frac{dY_t^*(y)}{Y_t^*(y)} = r_t dt - \langle \nu_t^*(Y_t^*(y)) + \tilde{\theta}_t, dW_t \rangle, \ Y_0^*(y) = y. \tag{0.0.10}$$

Ainsi le processus dual optimal ν^* dépend de la condition initiale y. Il en est de même pour la courbe des taux entre les dates T et $T+s$, donnée par:

$$R^{\nu^*(y)}(T,s) = -\frac{1}{s}\log \mathbb{E}^{\mathbb{P}}[\frac{1}{y}Y^{\nu^*}_{T,T+s}(y)|\mathcal{F}_T] = -\frac{1}{s}\log \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_T^{T+s} r_u du)|\mathcal{F}_T].$$

où la probabilité $\mathbb{Q}^{\nu^*(y)}$ s'écrit:

$$\frac{d\mathbb{Q}^{\nu^*(y)}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}\big(- \int_0^t \langle \tilde{\theta}_s + \nu_s^*(Y_s^*(y)), dW_s \rangle \big),$$

L'un des intérêts de cette généralisation est de faire dépendre la dynamique des taux du niveau général de richesse y dans l'économie. C'est une façon de déduire la structure de la courbe des taux mais c'est une moyenne. Pour vraiment utiliser toute l'information contenue dans $Y_t^*(y)$, nous considérons plutôt la dynamique des zéros-coupons dans ce cadre, par une approche similaire à cette des Chapitres 2 et 3 (sections 2.9.3 and 3.5), en utilisant le fait que le processus $Y_t^*(y)B^{\nu^*(y)}(t,T)$ est une martingale.

Chapter 1

Long-term interest rates

Contents

1.1 Motivations

Financial contracts written on mortality-related risks typically have a maturity up to 20 years, while, on the other hand, longevity-linked securities are typically characterized by a much longer maturity (40 years and beyond). In most of these contracts, there is an embedded interest rate risk. For the shorter time horizon (up to 20 years), the standard financial point of view can be used to hedge this risk as the interest rate market with such maturities is quite liquid. However, this is not the case anymore for longer maturities as the interest rate market becomes highly illiquid and the standard financial point of view cannot be easily extended. Due to this lack of information we are led to seek answers in the economic papers concerning long-term policy making.

In the economic papers, long term means typically a time horizon T between $T = 50$ and $T = 200$ years. Very often, the studies we consider are motivated by the financing in a world level of ecological stakes. The problem of how many points of growth we should sacrifice today in order to reduce the intensity of global warming is a typical example. A decision maker (an individual or a collective entity) has to decide if he invests today for projects which effect will only take place in the long term. Measures implemented today will only benefit for future generations. These benefits can be for example higher consumption, more wealth, a better health, a better state of the environnement, and they cause in any case an increase in the utility of the agents. In a recent past, such questions emerged, with the particularity of being related to the far-distant future. This is the case with the fight against global warming. In this case, this is not necesserily the same generation who has to make the investment and bear its cost, and who uses the benefits of this investment.

The main question which has emerged is: which discount rate should be used for the distant future? There is no reason to believe that one should discount all maturities at the same rate. Actually, Weitzman [Wei98] for example encouraged a decreasing discount rate. There is indeed a tendency to choose for the very long term a discount rate that is smaller than the one used to discount cash flows in the short term, for example a rate of 4 percent per year to discount cash flows up to 30 years and a rate of 2 percent for longer horizons.

Furthermore the fact that this is not the same generation that has the benefits of the investment and that has to bear its costs is particular and we wonder if it might have an impact on the discount rate. How much are current generations ready to pay for future generations? A classical assumption is that future generations will be richer than us. This assumption is realistic,

when one compares for example the last three hundred years, where the improvment has been significant. If we expect the same positive growth rate for the centuries to come, current generations should not be ready to pay to reduce the costs for future generations, who will be significantly richer than us. They could argue that implementing policies today (which will benefit future generations) costs money that could be used to improve the welfare of current generations (fighting pandemies, improve access to drinkable water, improve access to food and basic needs).
Furthermore, the fact that these long-term projects are spread out over several generations has been described as “non-commitment” ([Eke]). Indeed, it is not possible to commit future generations to follow the decisions that decision-makers have taken today. Future generations might change these decisions.
Another striking feature of long-term problems is the uncertainty that characterizes them. Indeed, the scarcity of natural resources, pandemies, or major political events of that would affect the whole world could cause a significant decrease in the growth rate. The possibility of such events shows the uncertainty concerning future growth.
In the case of such events, the previous assumption that future generations will be wealthier than current generations might no longer be relevant. If we start to think that such events could occurr then we will be more ready to invest for future generations. Then investing today for future generations is a precautionary measure. This is the precautionary effect.
The particular case of global warming, or the deterioration of the environment has been recently discussed in the Stern report and quantifies the possible effects of climate change and the cost of implementing policies today to fight against it. It considers the likelihood that the average temperature will increase by more than 5 Celsius degrees. The model also includes the probability of catastrophes if the temperature increases above this critical level. The catastrophes in question generate losses in the range of 5 to 20 percent of GDP (Gross Domestic Product). Combining all sources of uncertainty the Stern’s report leads to the concusion that the best estimate of losses in the year 2200 is 13.8% of GDP, with a 90% interval of confidence that the true loss will be between 2% and 35% of GDP. This result is however based on values of the climatic and economic parameters that are highly uncertain and depend on various assumptions. But these figures help to underline how huge is the uncertainty characterizing long term problematics.
Analyzing more precisely this high uncertainty, it can be split into two factors. First, the predictable events have a very high dispersion. What is called a predictable event is for example the fact that climate change will for

sure have a negative impact on our welfare. But we can not really quantify it. This is the high dispersion that appears in the conclusion of the Stern review (between 2% and 35% of GDP).
The other factor causing uncertainty is the fact that there could be non-predictable events. It is not possible to assign probabilities to these events. They are so rare that we do not know when or if they will occurr. But in any case they would be catastrophic, so it would be wise to take them into account. An example would be an increase of 10 meters of the sea level around the globe.
In the following we examine the state of the art of work concerning long term discounting and long term interest rates. Some of the factors previously mentioned, such as the wealth effect or the precautionary effect appear, and are more quantified, in the Ramsey Rule, which we present in the following section.

For long-term discounting the Ramsey Rule is a basic model and several of its extensions to draw conclusions concerning the discount rate to choose for long term environmental projects. Then we examine the case of uncertainty of the rate of return.

1.2 The Ramsey Rule

First we recall an approach inspired from the neoclassical economical theory and outlined by many authors: Gollier ([Gol], [Gol07b]), Ekeland ([Eke]) and Breeden.
One of the differences between this economic approach and the usual interest rate modelling approach in finance is that here the parameters governing the shape of the yield curve are not given exogenously. Here they are based on individual preferences. This makes sense when we recall that these interest rates can be interpreted as a function of the quantity of well-being that we should sacrifice today in order to finance some long-term projets.
In this framework, the economy is represented by the strategy of a representative agent, considered risk-averse and whose utility function on consumption u is assumed to be three times differentiable, increasing and concave. Moreover, the agent is assumed to behave as price taker. Here we call c_t the aggregate consumption and we denote by β its pure time preference parameter, i.e. β quantifies the agent preference of immediate goods versus future ones.
The agent might invest or not in a certain project. The particularity of this project is that its costs and benefits are generated over a long period of time

(as environmental projects). In this framework, we call R_t the sure rate of return of the project. That is, investing 1 euro at time 0 gives $e^{R_t t}$ at time t. Here $u'(c_0)$ represents the cost of reducing consumption by one monetary unit. Indeed, if the agent reduces consumption by ϵ at time 0, the loss in utility is equal to $\epsilon u'(c_0)$ (because $u(c_0-\epsilon)-u(c_0) \simeq -u'(c_0)\epsilon$). It is assumed that the cash flows of the investment are consumed at the end of the project. Hence this reduction of consumption allows for more accumulation and then for more consumption at time t, when the project is over. Consumption at time t is then increased by $\epsilon e^{R_t t}$, causing the expected utility to be increased by $\mathbb{E}[u'(c_t)]\epsilon e^{R_t t}$. These reallocations of consumption must leave the welfare unchanged along the optimal consumption path, i.e. the loss in utility at time 0 must be equal to the discounted increase of utility at time t. That is, at the equilibrium:

$$u'(c_0) = e^{-\beta t} e^{R_t t} \mathbb{E}[u'(c_t)].$$

Rewriting this condition, we obtain the classical consumption based pricing formula (Ramsey discount rate):

$$R_t = \beta - \frac{1}{t} \ln \frac{\mathbb{E}[u'(c_t)]}{u'(c_0)}. \tag{1.2.1}$$

Here the discount rate does not depend on the considered project. But it depends on the maturity t. We notice also that in this formula the consumption at time $t > 0$ might be random, but the consumption rate c_0 is deterministic at time 0. The spot rate R_t does not appears explcitly in this formula. The same formula holds for a stochastic or a deterministic r_t.

This intuitive way of obtaining (1.2.1) comes from macroeconomics [BF89].

A similar economic model has been introduced by [Bre89]. The derivation of the yield curve for far-distant maturities is induced from the maximization of the representative agent's intertemporal utility function on the aggregate consumption :

$$\max_{c \geq 0} \int_{t \geq 0} e^{-\delta t} u(c_t) dt$$

where c_t is the aggregate consumption, u the agent's utility function and δ his pure time preference parameter (i.e. δ quantifies the agent preference of immediate goods versus future ones). In the setting of deterministic rate and consumption, the optimal consumption is given by:

$$u'(\widehat{c}_t) e^{-\delta t} = u'(c_0) e^{-\int_0^t r_s ds}$$

where r_t is the spot rate and $e^{-\int_0^t r_s ds}$ the discount factor. This leads to the so-called "Ramsey rule"

$$\frac{1}{t}\int_0^t r_s ds = \delta - \frac{1}{t}\ln\frac{u'(\widehat{c_t})}{u'(c_0)}. \tag{1.2.2}$$

Adding uncertainty on the interest rate and the consumption, the maximization of the representative agent's utility function takes into account the budget constraint

$$\max_{c\geq 0}\int_{t\geq 0} e^{-\delta t}\mathbb{E}\left(u(\widetilde{c_t})\right)dt \quad \text{s.t. } \mathbb{E}\left(\int_{t\geq 0} e^{-\int_0^t \widetilde{r_s}ds}\widetilde{c_t}dt\right) \leq x_0.$$

The budget constraint expresses the initial wealth x_0 in the economy that allows to finance the consumption plan $\widetilde{c_t}$. The optimal consumption is given pathwise by:

$$\mathbb{E}(u'(\widehat{c_t}))e^{-\delta t} = u'(c_0)e^{-\int_0^t \widetilde{r_s}ds}.$$

The initial consumption c_0 is a function of the initial wealth x_0, given by the budget constraint. Note that the consumption is deterministic if and only if the interest rate is deterministic. The Ramsey rule can be extended in this stochastic framework :

$$R_0(t) := \frac{1}{t}\ln\mathbb{E}\left(e^{\int_0^t \widetilde{r_s}ds}\right) = \delta - \frac{1}{t}\ln\frac{\mathbb{E}(u'(\widehat{c_t}))}{u'(c_0)}, \tag{1.2.3}$$

The solution (1.2.1) can be developped using the second-order Taylor approximation and lead to the following equation ([Gol07b]):

$$R_t \simeq \beta + R(c_0)\frac{\mathbb{E}(c_t) - c_0}{tc_0} - \frac{1}{2}R(c_0)P(c_0)\frac{Var(c_t/c_0)}{t}, \tag{1.2.4}$$

where:

$$R(c) = -c\frac{u''(c)}{u'(c)} \text{ is the relative risk aversion parameter}$$

$$P(c) = -c\frac{u'''(c)}{u''(c)} \text{ is the relative prudence parameter.}$$

Developing this solution using the second-order Taylor approximation leads to the following equation (see [Gol07b]):

$$R_0(t) \simeq \delta + R(c_0)\frac{\mathbb{E}(\widehat{c_t}) - c_0}{tc_0} - \frac{1}{2}R(c_0)P(c_0)\frac{Var(\widehat{c_t}/c_0)}{t},$$

where $R(c) = -c\dfrac{u''(c)}{u'(c)}$ is the relative risk aversion parameter and $P(c) = -c\dfrac{u'''(c)}{u''(c)}$ is the relative prudence parameter.

The yield curve is now governed by three components, i.e. the three terms on right hand side of the equation above (1.2.4).
First of all, the representative agent is interested in goods that bring immediate satisfaction rather than those with the same effect later on the future. This appears in the preference parameter. This effect works in an additive manner with the second effect, the so-called "wealth effect". The individual prefers to consume rather then saving because he will be wealthier in the future (or future generations will be). This wealth effect increases in interest rate.
Finally, the "precautionary effect" raises when the future is uncertain and increases the representative agent's willingness to save. This precautionary effect goes opposite to the wealth effect and lowers the equilibrium interest rate. Those effects determine the optimal interest rates yield curve and shape when discounting far distant maturities.
For long horizons, we examine what the slope of the yield curve could be. The term structure is determined by two conflicting effects. A more distant future gives a larger expected consumption (wealth effect). Higher expectations about future income reduce the willingness to save and rise the equilibrium interest rate. But a more distant future gives also a larger uncertainty (precautionary effect). The willingness to save is increased by larger uncertainty about the future and this lowers the equilibrium interest rate. A decreasing curve is obtained if the wealth effect becomes more dominant compared to the precautionary effect. In the opposite case it is increasing.
It is important to notice here that many authors are in favor of a decreasing yield curve in the long term. For instance we refer to the declining discount rates presented by Weitzman [Wei98] or the work of Jouini and Napp [JN10].

The formula (1.2.1) is often combined with two other assumptions. The first is that the agent's utility function is chosen to be:

$$\begin{aligned} &u(c) = c^{1-\gamma}/(1-\gamma), \text{ pour } \gamma > 0, \gamma \neq 1 \\ &u(c) = \ln c \text{ for } \gamma = 1, \end{aligned}$$

where γ is the risk aversion. The other is that $c_t = c_0 e^{gt}$. In this particular case (1.2.1) becomes:

$$R_t = \beta + \gamma g.$$

This is the Ramsey Rule. The discount rate, net of the pure time preference parameter equals the product of growth rate of consumption by the index of relative risk aversion.

In several cases however, for example in [Gol], the consumption is chosen to be:

$$d \ln c_t = \mu dt + \sigma dW_t.$$

That is, in this framework, the consumption process is given exogenously and assumed to be lognormal. Then we obtain:

$$\begin{aligned} R_t &= \beta - \frac{1}{t} \ln \frac{\mathbb{E}[u'(c_t)]}{u'(c_0)} && (1.2.5) \\ &= \beta - \frac{1}{t} \ln \mathbb{E}[\exp(-\gamma(\ln c_t - \ln c_0))] && (1.2.6) \\ &= \beta + \gamma(\mu - 0.5\gamma\sigma^2), && (1.2.7) \end{aligned}$$

that is, with these assumptions, R_t is constant over time. This is the so-called extended Ramsey rule.
This expression shows that when relative risk aversion is constant and the growth rate of the economy (i.e. the log of the consumption) follows a stationary Brownian motion, the wealth effect and precautionary effect compensate each other and the yield curve is flat.
The extended Ramsey rule contains three terms, they correspond to the three effects on the interest rate mentioned above (present preference parameter, wealth effect and precautinary effect).
As a remark, we can say that $g = \mu + 0.5\sigma^2$ is the expected growth rate, then the equation can be rewritten:

$$R_t = \beta + \gamma g - 0.5\gamma(\gamma + 1)\sigma^2.$$

Then we provide some exemples of the values of the long term interest rates that are obtained within this framework.

- In the Stern Review ([Ste]), a logarithmic utility function ($\gamma = 0$), $\sigma = 0$, $\mu = 1.3\%$ (exponential consumption), $\beta = 0.1\%$, the discount rate R_t is constant and $R_t = 1.4\%$.
- In Weitzman (2007), with $\gamma = 2$, $\mu = 2\%$ and $\beta = 2\%$, we obtain $R_t = 6\%$.
- Gollier chooses $\beta = 0$ and $\gamma = 2$ (following Hall (1988)), $\mu = 2\%$, because the wealth per capita has been growing of approximately 2 percent per year during the twentieth century, and a volatility of 2 percent. Then he obtains $R_t = 3.92\%$. With this choice of parameters, the wealth effect is dominant, with $\gamma\mu = 4\%$ and the precautionary effect of $0.5\gamma^2\sigma^2 = 0.08\%$ has very little influence.

It is however difficult to imagine discounting with a constant discount rate over a long period of time. Therefore there are some extensions to the basic formula (see Section 1.3).

As we see, the Ramsey Rule (or even the simplified Ramsey Rule in the case where the discount rate is given by $R_t = \beta + \gamma g$) has caused controversy between economists concerning parameter values. In [LH], Hourcade et al. give possible numeric values for β, g and γ. The pure time preference parameter can be chosen between 0% and 4%. There is already controversy concerning its value. More precisely, Arrow [Arr95] and Manne [Man95] suggest $2\% \leq \beta \leq 4\%$. This value would reflect the current behaviour of the economic agents. Other authors suggest a lower pure time preference parameter, which would help for future generations. The growth rate g is estimated to be between 1% and 3% between years 2000 and 2100. The risk aversion parameter has been estimated by Arrow et al. [AC96]. It should be chosen in order to satisfy: $0.8 \leq \gamma \leq 1.6$.
However, all combinations of values of these parameters are not realistic, only some of them have an economic meaning. Moreover, Hourcade et al. [LH] have stated that the value of the pure time preference parameter is less important than assumptions about future growth, preferences or beliefs.

But in conclusion, the simplified Ramsey Rule is a very basic formula with few parameters. A way to refine the expression of the discount rate is to generalize the expression of the Ramsey Rule. This is the purpose of the following section.

1.3 Various extensions of the Ramsey rule

The Ramsey rule formula has several extensions. Gollier (2002) for instance relaxes the hypothesis on the form of the utility function. Also Weitzman (2004) and Gollier (2004) relax the hypothesis of a Brownian motion with constant coefficients.
In [Gol07b], the assumption on the dynamics of the consumption is relaxed by the assumption that:

$$d \ln c_t = \mu(s_t) + \sigma_c(s_t) dW_t,$$

where $ds_t = g(s_t)dt + \sigma_s(s_t)dW_t$.

It is crucial to take growth uncertainty into account. It is already a difficult task to predict the growth rate for the coming year. And in the long term the estimation of the growth is subject to enormous errors. Various disasters at a global scale might cause lower growth rates in the future, or even negative growth rates. On the other hand, improvement in technology which we are not capable of predicting today might also happen and cause a period of higher growth.

In the case of one permanent shock on the growth rate, Gollier assumes that the consumption is given by $c_t = c_0 e^{gt}$, where the growth g is random and takes a value g_1 with probability π and a value g_2 with a probability $1 - \pi$. Then the Ramsey Rule becomes:

$$R_t = \beta - \frac{1}{t}\ln(\pi e^{-\gamma g_1 t} + (1-\pi)e^{-\gamma g_2 t})$$

The time preference parameter β is chosen to be 0, and $\gamma = 2$. The random growth g has the distribution $(3\%, 2/3; 0\%, 1/3)$. This is, with a probability of two-third, the growth rate will be 3 percent, but with a probability 1/3 a certain shock happens, that will lower the growth to 0 percent, with no change in the future.
The example given above can be generalized: the uncertainty on the growth rate g can be modelled as follows (see this example in [Joub]). Let consider that g can take n different values g_i respectively with probability p_i. Then the Ramsey Rule becomes:

$$R_t = \beta - \frac{1}{t}\ln(\sum_{i=1}^{n} p_i e^{-\gamma g_i t}) - \frac{1}{2}\gamma^2\sigma^2.$$

Another refinement ([Gol07b]) is to choose a stochastic process for aggregate consumption with a drift that can take two possible values. A switch from one to the other can take place at each period with a small probability. If the drift rises, this takes into account the possibility of a technical revolution that suddenly increases the global consumption (such as the industrial revolution in the nineteenth century). The opposite switch represents the case of a shock in the opposite direction (limitation of natural resources,...).

In [Gol02], a random walk is used to model the growth and its changes, particularly to include the case of recession, but this case is solved in a discrete time framework.

1.4 The agent's beliefs

Another way to extend this model is to refine the modelling of the representative agent's beliefs.

Jouini [Joua], with a standard model of Pareto optimality, analyzes the behavioral properties of the representative agent. He models a world of N individuals, some of which are optimistic, other are pessimistic. Then the representative agent is represented as the aggregate behaviour of these agents: he is optimistic for some states of the world, and pessimistic for other states of the world. More precisely, the representative agent is afraid of very bad events, and on the contrary, he really desires good events.
In [Jou10], the case of an universe with N agents having heterogenous beliefs (each of them has a utility function $u^i(.)$ considered. The yield curve deduced from this equilibrium is also a Ramsey Rule -like formula.
This approach, though not specifically linked to the study of the long term, could contribute to refine the analysis: it is indeed very reasonable to assume that over a very long period of time the beliefs of the representative agent change, or are unknown. And it is underlined in [LH, HDH10], a good modeling of the agent's beliefs matters more than the time preference parameter.

1.5 Solving the Weitzman-Gollier puzzle

Here we present an example of controversy concerning the value of the discount rate, and the shape of the yield curve.
Weitzman [Wei98] has sugested that the lowest possible discount rate should be used for the long-distant future when future interest rates are uncertain. Indeed, using the present value gives a term structure of discount rates that decreases to the smallest value.
Gollier (2004) on the other hand has obtained an increasing term stucture of discount rates up to its largest value and he has recommended instead that the highest possible rate is used for long term discounting.
These two approaches are similar insomuch as they use intertemporal evaluation. But Weitzman uses the net present value (NPV) approach and Gollier uses the net future value (NFV) approach and the results they obtain are different. This is the so-called Weitzman-Gollier puzzle.
More precisely, we consider an investment that generates a sure payoff Φ at date t for each euro invested at date 0. Gollier considers θ the rate of return during the period. In the first case the agent wants to maximize the net present value of the future expected utility. Then the Net Present Value (NPV) is:

$$NPV = -1 + \Phi e^{-\theta t}.$$

The agent invests 1 euro at date 0 and he borrows $\Phi e^{-\theta t}$ also at date $t = 0$. This means that he bears all the cost. There are no other net payoffs along the lifetime of the project, as shows the following arbitrage argument:

$$\begin{array}{ccc} & 0 & t \\ Investment & -1 & \Phi \\ Loan & \Phi e^{-\theta t} & -\Phi \\ & NPV & 0 \end{array}$$

The agent invests if $NPV \geq 0$.
Alternatively the agent could transfer the costs to future generations and borrow 1 euro at date 0 to cover the cost of his investment. Then we get:

$$\begin{array}{ccc} & 0 & t \\ Investment & -1 & \Phi \\ Loan & 1 & -e^{\theta t} \\ & 0 & NFV \end{array}$$

where NFV is the Net Future Value:

$$NFV = -e^{\theta t} + \Phi.$$

The agent invests if $NFV \geq 0$.
In this case $NFV = e^{\theta t} NPV$ and making a choice by considering the Net Present Value or the Net Future Value is equivalent. However if there is uncertainty on θ this is no longer the case.
More precisely, they consider that $\tilde{\theta}$ the interest rate that will prevail betweem 0 and t is uncertain. $\tilde{\theta}$ is a random variable taking values in $\mathbb{R}$ and describing the uncertainty of the interest rate. θ is a realisation of $\tilde{\theta}$. In this basic framework, shortly before time 0 where the project starts, the interest rate in unknown but the agent must however take his decision. If he decides to borrow a certain amount of money, this loan will be made at the uncertain rate $\tilde{\theta}$. But just after $t = 0$ the value $\tilde{\theta}$ is revealed and permanently set to $\tilde{\theta} = \theta$ and there is no more uncertainty. For Weitzman, the investor invests in a project if the project has a positive expected Net Present Value (obtained in the same way as the net present value in the non random case):

$$ENPV = -1 + \Phi \mathbb{E}[e^{-\tilde{\theta} t}].$$

This is equivalent to discounting with a certainty-equivalent discount rate R_t^p, which gives a net present value of:

$$-1 + \Phi e^{-R_t^p t}.$$

From these two expressions, he obtains a discount rate verifying $\mathbb{E}[e^{-\tilde{\theta} t}] = e^{-R_t^p t}$ and finally:

$$R_t^p = -\frac{1}{t} \ln \mathbb{E}[e^{-\tilde{\theta} t}]. \tag{1.5.1}$$

Hence R_t^p is decreasing with t.
Gollier (2004) considers that an agent invests in a projet if the expected Net

Future Value is positive. This expected Net Future Value is expressed as:

$$ENFV = -\mathbb{E}[e^{\tilde{\theta}t}] + \Phi.$$

Similarly, the certainty equivalent discount rate R_t^f produces a net future value of $-e^{R_t^f t} + \Phi$ and equaling the two expressions gives a certainty equivalent rate of:

$$R_t^f = \frac{1}{t} \ln \mathbb{E}[e^{\tilde{\theta}t}]. \tag{1.5.2}$$

Hence R_t^f is increasing with t.

These two approaches (Net Present Value and Net Future Value) give different shapes for the interest rate curve, and lead to opposite conclusions. This is the Weitzman-Gollier puzzle.

Some papers other have implemented ways to solve this puzzle and to reconcile the two approaches, for example [BS08]. For them, investing at the discount rate R_{bs} has the same utility as the expected utility of investing at the uncertain rate of return of capital $\tilde{\theta}$: $u(e^{R_{bs}t}) = \mathbb{E}[u(e^{\tilde{\theta}t})]$. Then if the risk-aversion is large enough, R_{bs} is decreasing with t. In this framework discount rates are decreasing for large maturities.
In Hepburn and Groom ([HG07]), the chosen framework is closer to a continuous time approach. Indeed, the cashflows of the project are represented using the following expression:

$$N_t = -\delta_0 + \Phi\delta_T,$$

where δ_t is a Dirac at time t. And then the Expected Net Present Value is the expectation of the integral of this cash flow.

$$ENPV = \mathbb{E}\left[\int_0^T N_t e^{\tilde{\theta}t} dt\right].$$

This gives the same value of the net present expected value.

Gollier explains the differences between the present value and the future value approaches by the fact that in one case all the risk is at date 0 and in the other case all the risk is at date t. He shows that these approaches are equivalent and can be solved by introducing utilities and changing the probability measure. They are also equivalent to the so-called Ramsey discount rate extended to uncertainty.
From now on we denote the consumption $\tilde{c}_t$ and we consider that it is stochastic, including $\tilde{c}_0$ at time 0. What Gollier calls the Ramsey discount rate

extended to uncertainty is:

$$R_t = \beta - \frac{1}{t} \ln \frac{\mathbb{E}[u'(\tilde{c}_t)]}{\mathbb{E}[u'(\tilde{c}_0)]}.$$

Then he considers the Net Present Value and Net Future Value in this framework. We call $\hat{R}_t^p$ the certainty equivalent discount rate in this case. If the investor invests ϵ at time 0, the certain payoff at time t is $\epsilon e^{\hat{R}_t^p t}$. If in addition to this the investor borrows $\epsilon e^{\hat{R}_t^p t} e^{-\tilde{\theta} t}$ at time 0, the sum of the cash flows at time t is zero and so is the increase in utility. Therefore the expected utility loss at time 0 is: $\mathbb{E}[\epsilon u'(\tilde{c}_0)(-1 + e^{\hat{R}_t^p t} e^{-\tilde{\theta}} t)] = 0$. This expression is equal to the increase in utility at time t discounted by the present preference parameter, that is 0. This gives the discount rate $\hat{R}_t^p$:

$$\hat{R}_t^p = -\frac{1}{t} \ln \frac{\mathbb{E}[u'(\tilde{c}_0) e^{-\tilde{\theta} t}]}{\mathbb{E}[u'(\tilde{c}_0)]} \tag{1.5.3}$$

For Gollier, this is a generalization of the Net Present Value discount rate (1.5.1), except that here the expectation of $e^{-\tilde{\theta} t}$ is replaced by a weighted expectation of $e^{-\tilde{\theta} t}$. The weight in question is:

$$\frac{u'(\tilde{c}_0)}{\mathbb{E}[u'(\tilde{c}_0)]}.$$

Gollier calls this a risk-neutral expectation operator.
Similarly, using the Net Forward Value approach, he obtains a discount rate $\hat{R}_t^f$ of:

$$\hat{R}_t^f = \frac{1}{t} \ln \frac{\mathbb{E}[u'(\tilde{c}_t) e^{-\tilde{\theta} t}]}{\mathbb{E}[u'(\tilde{c}_t)]} \tag{1.5.4}$$

This is also a generalization of (equation (1.5.2)), except that the expectation of $e^{-\tilde{\theta} t}$ is replaced by a weighted expectation with weight $\frac{u'(\tilde{c}_t)}{\mathbb{E}[u'(\tilde{c}_t)]}$.
Gollier ([Gol09c]) puts together these two approaches.
Conditionally to $\tilde{\theta} = \theta$, the representative agent maximizes his discounted utility from consumption (with discrete dates):

$$\max_c \sum_{c_0, c_1, \ldots, c_t} e^{-\beta t} u(c_t),$$

under the constraint, at each time period:

$$K_t = e^{\theta} K_{t-1} - \tilde{c}_{t-1} \geq 0,$$

where K_t is the capital of the agent, for all dates $t = 1, \ldots, T$. The equation obtained for the optimal consumption process is:

$$u'(c_t^{(\theta)}) = \xi(\theta)e^{(\beta-\theta)t}, \tag{1.5.5}$$

where $\xi(\theta)$ is the Lagrangian multiplier associated to the intertemporal budget constraint. Then this is also equal to:

$$u'(c_t^{(\theta)}) = u'(c_0^{(\theta)})e^{(\beta-\theta)t},$$

From (1.5.5) he deduces a non conditional relation:

$$u'(c_t^{(\tilde{\theta})}) = \xi(\tilde{\theta})e^{(\beta-\tilde{\theta})t},$$

Using this equation and replacing in (1.5.3) and (1.5.4) it shows that:

$$\hat{R}_t^p = \hat{R}_t^f = R_t.$$

1.6 Taking different beliefs concerning the discount rate into account

As we see, there are several controversies concerning the value of the ong term growth rate. The work of Jouini et al. [JN10, JMN10] takes into account the anticipations of the agents. In their model, there are agents with different beliefs concerning the growth rate of the economy g_i, and they have also different pure time preference parameters. Each of them believes in a discount rate $R_i(t)$. An equilibrium is established. In their model, the yield curve decreases, and for a longer maturity $R(t) = R_{i_0}(t)$: for the long term, the interest rate of the most pessimistic agent is chosen, and it corresponds to the lowest rate.

1.7 The impact of environment

As we have seen previously, one of the reasons why long-term maturities have become an issue is because they appear in the problematics concerning the environment. One of the extensions of the Ramsey Rule framework is the introduction of an environmental good. Gollier [Gol] examines how much wealth we are ready to sacrifice today in order to improve the future quality of the environment. The maximization problem of the representative agent is now the maximization of the expected discounted utility from consumption and environment:

$$\max \mathbb{E}\left[\int e^{-\delta t} U(x_{1,t}, x_{2,t}) dt\right].$$

This gives two different yield curves. The environmental factor is also taken into account by Ekeland ([Eke]), [Gue04] and [Gol09a].
More generally, it underlines the fact, mentioned in [LH] that the nature of what we give to the following generations has an impact on the yield curve. Indeed, what do we leave to following generations: is it just cash? or a good environmental quality? or advanced knowledge so that they can adapt better to new environmental situations? The nature of our bequest has its importance. It is thus interesting to distinguish between utility from consumption, wealth, an environmental good, or event a certain amount of knowledge, although modelling these quantities becomes really complex.

1.8 Conclusion

Many other extensions of the basic Ramsey Rule model (1.2.1) have been considered. Another extension is the use of sophisticated model for consumption. For example, Scheinkman uses a pure Levy process to model the consumption and derive the optimal interest rates yield curve on a Markov environment.
Several other factors could be reasonably considered such as an unknown probability distribution about the growth of the economy, extreme events, and habit formation for the agent. By taking some of these factors into account, it is possible to improve the mathematical representations of uncertainty in the long term.

Chapter 2

Growth Optimal Portfolio, utility maximization and interest rates in a complete market

Contents

The Growth Optimal Portfolio is a particularly robust portfolio over long periods of time. It is also possible to use the Growth Optimal Portfolio as a numeraire or benchmark for pricing zero coupons. Hence it seems to be a useful tool for our purpose of studying long term interest rates.
In this chapter, we present the work of Platen and Heath [PH06] concerning the properties of the Growth Optimal Portfolio.
We introduce also our framework: a complete financial market with N risky assets, and we recall expected utility maximization problems and their solutions, along the lines of Karatzas et al. [KS98].
Then we underline important properties of the Growth Optimal Portfolio (or GOP): it is the inverse of the state price density process, it is strongly related to the expression of the optimal consumption path, and finally we deduce the Ramsey rule in complete markets.

2.1 The Growth Optimal Portfolio

In [PH06], Heath and Platen present the Growth Optimal Portfolio or GOP. We recall some of its properties in this introductory sections. The Growth Optimal Portfolio is a remarkably stable portfolio, which exists in any reasonable financial market. The GOP is obtained by answering the question: in which numeraire should the payoff be expressed to apply an expectation under the real world probability measure? This means that one has to find a strictly positive process which, when used as a numeraire or benchmark, generates derivative price processes that are martingales with respect to the real world (or historical) probability measure.
The so-called Benchmark Approach, also presented in [PH06] uses the GOP as a benchmark, reference unit or numeraire and the real world (or historical) probability measure is the pricing measure. Derivative pricing formulas can be written in terms of the real world probability measure.

In this part we recall several characterizations of the GOP. First of all the GOP is the portfolio that maximizes the growth rate over any time horizon (the growth rate being the drift of the logarithm of the portfolio value).
Kelly [Kel56] has also characterized the GOP as the portfolio maximizing the expected log-utility from terminal wealth over all strictly positive portfolios.
The Growth Optimal Portfolio has the maximal growth rate over any time horizon $T > 0$ and we think that it can be used to accurately describe our problem concerning long term interest rates.

Another advantage of the GOP is that it has a quite general form, it does not require many assumptions on its coefficients, and it can be approximated by a world stock index and therefore linked to real world data. Thus, even if one may never find a perfectly accurate model for the stock market dynamics, a diversified world stock index, approximates well the GOP.

2.2 Notations

In this section we describe a complete financial market, such as in [PH06], [KS98] and give the properties of the GOP, which have been studied in [PH06] and several other references.

In the following, for any two vectors $a = (a^1, \dots, a^j, \dots, a^N)^{\mathbf{T}}$ and $b = (b^1, \dots, b^j, \dots, b^N)^{\mathbf{T}}$ of size N, we denote by:

$$a^{\mathbf{T}}.b = \langle a, b \rangle = \sum_{j=1}^{n} a^j b^j,$$

their scalar product and

$$||b|| = \sqrt{\sum_{j=1}^{N} (b^j)^2}.$$

We denote by $\mathbf{1}_N$ the vector of size N where each entry is equal to one.
In this framework, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration of $\mathcal{F}$ satisfying the usual conditions that is, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$. For any $0 \leq t \leq T$, $\mathcal{F}_t$ denotes the information available in the market at time t.

We consider a continuous financial market $\mathcal{M}$, which is assumed to be a complete market. This financial market consists of:

- a time horizon $T > 0$,
- an N dimensional $\mathbb{F}$-Brownian motion $(W_t)_{0 \leq t \leq T}$.
- the spot rate $r(.)$, which is a positive, progressively measurable process, satisfying $\int_0^T r_s ds < +\infty$ almost surely,
- a progressively measurable rate of return process $b(.)$, which is a $N \times 1$ column vector such that $b_t = (b_t^1, \dots, b_t^N)^{\mathbf{T}}$ satisfies $\int_0^T ||b_s|| ds < \infty$ a.s.

- a progressively measurable $N \times N$ matrix-valued volatility process $\sigma(.)$ satisfying $\sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^T (\sigma_t^{jk})^2 dt < \infty$ a.s.,
- and a N by 1 column vector of positive constants, representing the initial prices of the assets: $S(0) = (S_0^1, \dots, S_0^j, \dots, S_0^N)^{\mathbf{T}}$.

In this section, the market $\mathcal{M}$ is a complete market. In the financial market we described the number of risky assets is N, that is the same as the number of sources of uncertainty N (and as the dimension of the underlying Brownian motion $(W_t)_{0 \leq t \leq T}$).
In the following we denote this market:

$$\mathcal{M} = (r(.), b(.), \sigma(.), S(0)).$$

We remind that the dynamics of the riskless asset S^0 is:

$$dS_t^0 = r_t S_t^0 dt, \ S_0^0 = 1,$$

and the dynamics of the risky asset S^j, for $j \in 1, \dots, N$ are:

$$dS_t^j = S_t^j \big(b_t^j dt + \sum_{k=1}^{N} \sigma_t^{jk} dW_t^k \big), \tag{2.2.1}$$

with initial value S_0^j.
In this market where no asset is redundant we add the following classical assumption:

Assumption 1 *The volatility matrix $(\sigma_t^{j,k})_{1 \leq j,k \leq n}$ is invertible for Lebesgue-almost every $t \in [0, T]$.*

The risk premium process is $\theta : [0, T] \times \Omega \to \mathbb{R}^N$ is the vector of size N satisfying, for $t \geq 0$:

$$\theta_t = \sigma_t^{-1}(b_t - r_t \mathbf{1}_N), \ 0 \leq t \leq T,$$

where we denote by $\mathbf{1}_N$ the N-dimensional vector whose every component is one, and where σ_t^{-1} is the inverse of the volatility matrix. We assume that the $\mathbb{F}$-progressively measurable process $\theta(.)$ satisfies $\int_0^T ||\theta_s||^2 ds < +\infty$.
We can rewrite the SDE for the j-th risky asset (2.2.1) as:

$$dS_t^j = S_t^j \big(r_t dt + \langle (\sigma^{\mathbf{T}})_t^j, \theta_t dt + dW_t \rangle \big), \tag{2.2.2}$$

where $\sigma^{\mathbf{T}}$ is the transpose of the matrix σ and $(\sigma^{\mathbf{T}})^j$ is its j-th column vector.
We call:

$$||\theta_t|| = \sqrt{\sum_{k=1}^{N} (\theta_t^k)^2},$$

for $t \in [0, T]$ the total risk premium.

Definition 2.2.1 *We define by Z_t^0 the exponential local martingale:*

$$Z_t^0 = exp\left(-\int_0^t \langle \theta_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\theta_s||^2 ds \right). \qquad (2.2.3)$$

If we add the Novikov condition

$$\mathbb{E}\left[exp\big(\frac{1}{2}\int_0^T ||\theta_t||^2 dt\big) \right] < +\infty,$$

then the local martingale Z_t^0 is a martingale. It is the solution of the SDE:

$$dZ_t^0 = -Z_t^0 \langle \theta_t, dW_t \rangle, \ Z_0^0 = 1.$$

We define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by:

$$\mathbb{Q}(A) = \mathbb{E}[Z_T^0 \mathbf{1}_A], \ \forall A \in \mathcal{F}_T.$$

The process Z_t^0 is the probability density of the probability measure $\mathbb{Q}$ relatively to the historical probability measure $\mathbb{P}$. According to Girsanov's theorem, the process:

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s ds, \ t \in [0, T],$$

is a N-dimensional Brownian motion under $\mathbb{Q}$, relative to the filtration $\mathbb{F}$. Throughout Sections 2.1 to 2.7 of this chapter we assume the existence of the risk-neutral probability $\mathbb{Q}$.

Definition 2.2.2 *The state price density process $H^0(.)$ is defined for all $t \in [0, T]$ by:*

$$H_t^0 = \frac{Z_t^0}{S_t^0}.$$

In the continuous financial market $\mathcal{M}$ the investor forms portfolios containing a certain number of units of risky assets (and of the riskless asset). We consider the strategy ϕ, the predictable S-integrable process such that at time $t \in [0, T]$, $\phi_t = (\phi_t^0, \ldots, \phi_t^N)^{\mathbf{T}} \in \mathbb{R}^{N+1}$. For any $j \in \{0, 1, \ldots, N\}$, ϕ_t^j is the number of units of the j-th asset held at time t in the portfolio $S^\phi(.)$, which has value at time t:

$$S_t^\phi = \sum_{j=0}^N \phi_t^j S_t^j.$$

In the following we assume that all strategies $\phi(.)$ and porfolios S_t^ϕ satisfy the self-financing condition:

$$dS_t^\phi = \sum_{j=0}^{N} \phi_t^j dS_t^j.$$

This gives, by replacing in (2.2.2) and hence using the parametrization in terms of the risk premium vector that the value S_t^ϕ of a portfolio satisfies the SDE:

$$dS_t^\phi = S_t^\phi r_t dt + \sum_{k=1}^{N}\sum_{j=1}^{N} \phi_t^j S_t^j \sigma_t^{jk}(\theta_t^k dt + dW_t^k), \qquad (2.2.4)$$

for any $t \in [0,T]$.

Definition 2.2.3 *In the following we call $\mathfrak{N}^+$ the set of all strictly positive portfolios.*

For a given self-financing strategy ϕ, we denote by $\pi_t^j = \phi_t^j \frac{S_t^j}{S_t^\phi}$, for $t \in [0,T]$ and $j = 1,\dots,N$ the fraction of S_t^ϕ invested in the asset j at time t.
The coefficients π_t^j are positive or negative and such that $\sum_{j=0}^{N} \pi_t^j = 1$. We call $\pi_t = (\pi_t^1,\dots,\pi_t^N)^{\mathbf{T}}$ the column vector of size N having the π_t^j as components. Then the dynamics of a portfolio value S_t^π at time t can be rewritten in the form:

$$dS_t^\pi = S_t^\pi\big(r_t dt + \pi_t^{\mathbf{T}}\sigma_t(\theta_t dt + dW_t)\big). \qquad (2.2.5)$$

An important quantity in the equation above is the vector $\sigma_t^{\mathbf{T}}\pi_t$. For the sake of simplicity, we define the N by 1 column vector:

$$\kappa_t := \sigma_t^{\mathbf{T}}\pi_t.$$

Thus portfolios are more naturally parametrized in terms of κ. In the following we denote them by $S^\kappa(.)$. Then is dynamics of the portfolio S^κ is a solution of:

$$dS_t^\kappa = S_t^\kappa\big(r_t dt + \langle\kappa_t, \theta_t dt + dW_t\rangle\big). \qquad (2.2.6)$$

We define the porfolio growth rate g_t^κ the drift in the SDE giving the dynamics of $\ln S_t^\kappa$. Of course, the growth rate defined here should not be confused with $\tilde{g}$, the long term growth rate.

2.3 Dynamics and properties of the Growth Optimal Portfolio

Here we provide some background concerning the benchmark approach of [PH06]. First we give definitions for the Growth Optimal Portfolio (or GOP), which is considered to be the best performing portfolio in various ways. It was originally discovered by Kelly [Kel56] and then later it was studied in a financial framework by several authors such as Long [Lon90], El Karoui, Geman and Rochet [EGR95], Artzner [Art97], Becherer [Bec01], Bajeux-Besnainou and Portait [BBP97]. These definitions, and the various characterizations of the GOP highlight the role that it plays in finance.
The content of this section is based on [PH06] and others references by the same authors, especially: [Pla06, Pla04a, Pla04b, Pla09, PR09].

Definition 2.3.1 *A GOP or Growth Optimal Portfolio is the strictly positive portfolio process S^θ that maximizes the portfolio growth rate g_t^κ for all t. This means that for all $t \geq 0$ and all strictly positive portfolios $S^\kappa \in \mathfrak{N}^+$, the growth rate of the GOP g_t^θ satisfies the inequality*

$$g_t^\theta \geq g_t^\kappa \tag{2.3.1}$$

almost surely.

Another characterization of the GOP comes from Kelly [Kel56] and Long [Lon90]:

Definition 2.3.2 *The Growth Optimal Portfolio is the portfolio that maximizes the expected logarithmic utility from terminal wealth, that is $\mathbb{E}[\ln(S_T^\kappa)]$, for all $T > 0$, on all strictly positive portfolios $S^\kappa \in \mathfrak{N}^+$, the set of all strictly positive portfolios.*

We use Definition 2.3.1 to express the dynamics of the GOP. The growth rate g_t^κ of a portfolio is defined as the drift of the SDE of the logarithm of the portfolio S^κ. The form of the GOP is uniquely determined when maximizing the growth rate g_t^κ. Hence we apply Itô formula to obtain the SDE for the dynamics of $\ln(S_t^\kappa)$:

$$d\ln(S_t^\kappa) = g_t^\kappa dt + \langle \kappa_t, dW_t \rangle, \tag{2.3.2}$$

where the growth rate g_t^κ is defined as the drift of the SDE of the logarithm of the portfolio S^κ:

$$g_t^\kappa = r_t + \big(\langle \kappa_t, \theta_t \rangle - \frac{1}{2}||\kappa_t||^2\big) \tag{2.3.3}$$

We maximize this growth rate over all strictly positive portfolios. For this purpose, we derive (2.3.3) relatively to the κ_t^j, for all $j = 1, \ldots, N$. First order conditions give the optimal vector κ_t, for $t \in [0, T]$:

$$\kappa_t = \theta_t.$$

Replacing in 2.3.3 we deduce the dynamics of the GOP:

$$dS_t^\theta = S_t^\theta (r_t dt + \langle \theta_t, dW_t \rangle + ||\theta_t||^2 dt). \tag{2.3.4}$$

Thus one can check that the expression of the value of the GOP at time t, for all $0 \leq t \leq T$ is:

$$S_t^\theta = S_0^\theta \exp\left(\int_0^t r_s ds + \int_0^t \langle \theta_s, dW_s \rangle + \frac{1}{2} \int_0^t ||\theta_s||^2 dt \right). \tag{2.3.5}$$

The GOP is uniquely determined up to its initial value S_0^θ and its dynamics is characterized by the risk premium vector $\theta_t = (\theta_t^1, \ldots, \theta_t^n)^{\mathbf{T}}$ and the spot rate r_t, for any $t \in [0, T]$. This first definition of the GOP is given in a pathwise sense and does not involve utility functions or expectations.

Example 2.3.1 *A particularly simple example is the case where the market is composed of one riskless asset $S^0(.)$ with constant spot rate r and one risky asset $S(.)$ following a Black-Scholes dynamics:*

$$dS_t^0 = rS_t^0 dt, S_0^0 = 1,$$

$$dS_t = S_t(bdt + \sigma dW_t),$$

where $b \in \mathbb{R}$, $\sigma > 0$, $S_0 > 0$ and $(W_t)_{0 \leq t \leq T}$ is a one-dimensional $\mathbb{P}$-Brownian motion.
We call θ the risk premium:

$$\theta = \frac{b - r}{\sigma}.$$

For such a model with only one risky asset and constant parameters, the growth rate and the fraction invested in the risky asset are constant in time, and we denote them respectively g^κ and κ. Using equation (2.3.3), we have the expression of the growth rate as a function of κ:

$$g^\kappa = r + \kappa\theta - \frac{1}{2}\kappa^2. \tag{2.3.6}$$

In Figure (2.1), we represent the growth rate g^π as a function of the fraction π. The choice of parameters is in this case $r = 0.05$, $b = 0.07$ and $\sigma = 0.2$.

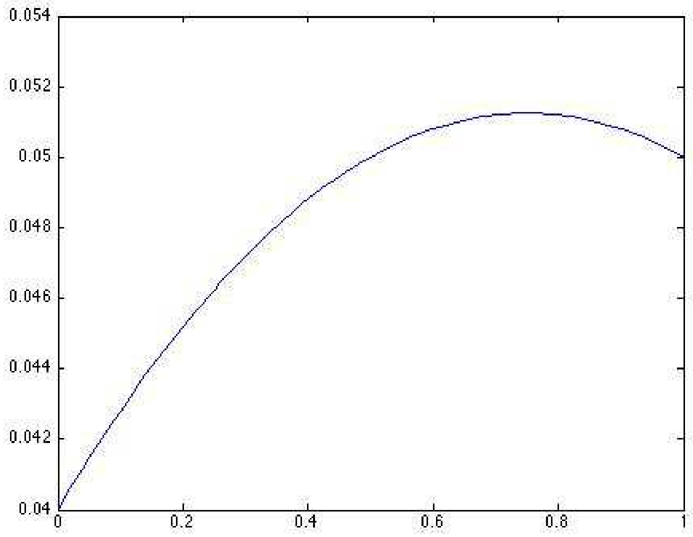


Figure 2.1: Growth rate g^π in a Black-Scholes model as a function of the fraction of wealth π invested in the risky asset

The GOP is the portfolio with maximal growth rate. In Figure (2.1), the maximum of the growth rate is attained for $\kappa = \theta$, that is:

$$\pi = \frac{\theta}{\sigma} = 0.75. \tag{2.3.7}$$

Replacing (2.3.7) into (2.3.6) gives the growth rate of the GOP:

$$g^\theta = r + \frac{\theta^2}{2}.$$

Replacing (2.3.7) into (2.3.5) the dynamics of the GOP:

$$S_t^\theta = S_0^\theta exp\big((r + \frac{\theta^2}{2})t + \theta W_t\big).$$

And the long-term growth rate of the GOP, which we recall is defined as $\tilde{g}^\theta = \limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_T^\theta}{S_0^\theta}\right)$, is also:

$$\tilde{g}^\theta = r + \frac{\theta^2}{2}.$$

If we choose in particular $S_0 = S_0^\theta = 1$, $\sigma = 0.2$, $r = 0.05$ et $b = 0.07$, and a horizon of $T = 10$ years, we have the following evolution for the path of the risky asset, riskless asset and GOP (Figure 2.2):

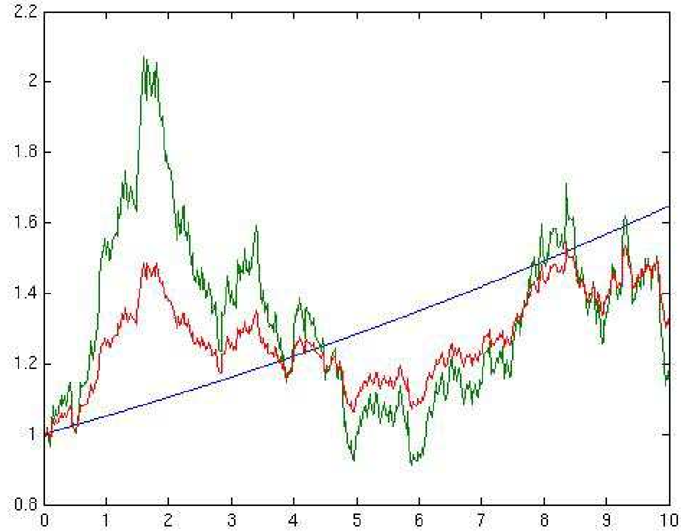


Figure 2.2: Riskless asset S_t^0 (blue), risky asset S_t (green) and GOP S_t^θ (red)

With the study of long term problems in hindsight we seek to observe an example of paths for a horizon of, for example, $T = 100$ years (Figure 2.3)

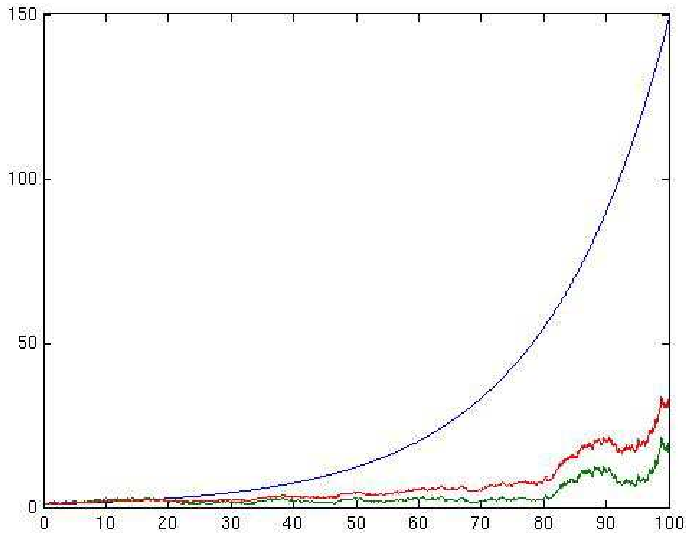


Figure 2.3: Riskless asset S_t^0 (blue), risky asset S_t (green) and GOP S_t^θ (red)

Proposition 2.3.1 *We have the following relation between the GOP and the* *state price density process H_t^0 previously defined:*

$$\frac{S_0^\theta}{S_t^\theta} = H_t^0, \tag{2.3.8}$$

for all $0 \le t \le T$.

Proof. Using 2.3.5 and 2.2.2, it is sufficient to write:

$$\frac{S_0^\theta}{S_t^\theta} = \exp\big(- \int_0^t r_s ds - \int_0^t \langle \theta_s, dW_s \rangle - \frac{1}{2} \int_0^t ||\theta_s||^2 ds \big) = H_t^0.$$

■

Hence the GOP S_t^θ is equal to the inverse of the state price density process up to one constant, which is the initial value S_0^θ. Thus in order to get rid of that constant we use a normalized expression of the GOP.

Definition 2.3.3 *The normalized GOP, which we denote $G^*(.)$ is such that for any $t \in [0, T]$:*

$$G_t^* = \frac{S_t^\theta}{S_0^\theta},$$

and $G_0^* = 1$. Because later we will focus on more dynamic problems, we also define, for all $0 \le s \le t \le T$:

$$G_{s,t}^* = \frac{S_t^\theta}{S_s^\theta},$$

and similarly $G_{0,0}^* = 1$. It is also sometimes called the numeraire portfolio.

This normalized GOP satisfies the equation:

$$\frac{dG_t^*}{G_t^*} = r_t dt + \langle \theta_t, dW_t \rangle + ||\theta_t||^2 dt, \;\; G_0^* = 1.$$

We call a discounted portfolio, a porfolio $S^\kappa(.)$ divided by the riskless asset, that is:

$$\bar{S}_t^\kappa = \frac{S_t^\kappa}{S_t^0}.$$

We also denote by $\bar{S}_t^\theta$ the discounted value of the GOP, that is S_t^θ divided by the riskless asset S_t^0:

$$\bar{S}_t^\theta = \frac{S_t^\theta}{S_t^0}.$$

The discounted GOP satisfies the SDE:

$$d\bar{S}_t^\theta = \bar{S}_t^\theta(\langle\theta_t, dW_t\rangle + ||\theta_t||^2 dt). \tag{2.3.9}$$

And its expression is:

$$\bar{S}_t^\theta = \bar{S}_0^\theta \exp\big(\int_0^t \langle\theta_s, dW_s\rangle + \frac{1}{2}\int_0^t ||\theta_s||^2 dt\big). \tag{2.3.10}$$

By discounting the GOP we have disconnected the impact of the spot rate and of the risk premium on the dynamics of the GOP. For the dynamics of the discounted GOP, only the dynamics of the risk premium has to be taken into account.
Now we examine what becomes the dynamics of portfolios when the GOP is used as a reference unit or numeraire. In the following, prices that are expressed in units of the GOP are called benchmarked prices.

Definition 2.3.4 *For any portfolio $S^\kappa(.)$ satisfying the assumptions mentioned previously, we define the benchmarked portfolio $\hat{S}^\kappa(.)$, for $t \in [0,T]$ by:*

$$\hat{S}_t^\kappa = \frac{S_t^\kappa}{S_t^\theta}.$$

Then, using Itô's formula, the dynamics of the GOP (2.3.5) and of a portfolio $S^\kappa(.)$ (2.2.6), a benchmarked portfolio satisfies the following SDE, for all $t \in [0,T]$:

$$\begin{aligned} d\hat{S}_t^\kappa &= \frac{dS_t^\kappa}{S_t^\theta} - \frac{S_t^\kappa}{(S_t^\theta)^2}dS_t^\theta + \frac{S_t^\kappa}{(S_t^\theta)^3}\langle dS_t^\theta, dS_t^\theta\rangle - \frac{1}{(S_t^\theta)^2}\langle dS_t^\kappa, dS_t^\theta\rangle \\ &= \hat{S}_t^\kappa\langle\kappa_t - \theta_t, dW_t\rangle. \end{aligned}$$

Here we see why the choice of the GOP as a numeraire has advantages over other alternatives. This SDE is driftless: any benchmarked portfolio $\hat{S}^\kappa(.)$ is a local martingale. Because it is nonnegative, any benchmarked porfolio is an $(\mathbb{F}, \mathbb{P})$-supermartingale. This means that the current benchmarked value of the portfolio is always greater or equal than its expected future benchmarked value:

$$\hat{S}_t^\kappa \geq \mathbb{E}[\hat{S}_s^\kappa|\mathcal{F}_t],\ 0 \leq t \leq s \leq T.$$

Under appropriate conditions on the coefficients, it is a martingale under the historical (or real world) probability $\mathbb{P}$.

2.4 Maximizing utility from consumption: framework

In the first chapter we have seen how in the economic theory, we obtain the Ramsey Rule by solving an expected utility from consumption maximization problem.
We would like to generalize this framework by taking into account the existence of a financial market. For this purpose we use the tools developed by [KS98] and other authors. More precisely, we maximize the expected utility from consumption and terminal wealth of a representative agent, who consumes a fraction of his wealth. Thus he seeks to solve the following maximization problem:

$$\sup_c \mathbb{E}\left[\int_0^T U(t, c_t)dt\right],$$

under a certain budget constraint.
We start with the most simple framework: a complete financial market. The financial market here is the complete market $\mathcal{M}(r(.), b(.), \sigma(.), S(0))$ described previously (in Section 2.2).

Besides, in the previous paragraphs, we have presented the Growth Optimal Portfolio, or GOP and its properties. Due to its properties, the GOP is a good candidate for our study of long term problems, thus we would like to take it into account.
One of these stricking properties is that in a complete market the GOP can be expressed as the inverse (up to a constant) of the state price density process H_t^0. The solutions of the classical expected utility from consumption and terminal wealth maximization problem, that is the optimal consumption path and the composition of the optimal portfolio can be expressed as functions of H_t^0. Thus we show the link between the optimal consumption and the GOP.
Finally we show that in the framework of a complete market, it is possible to deduce the Ramsey Rule from the expression of the trajectory of the optimal consumption.

2.4.1 Consumption, portfolio and wealth

We are interested in studying the global economy, in particular interest rates. Thus we consider a representative agent, representing the economy. His initial endowment, or initial wealth, is $x > 0$. Until now, we have only considered portfolios S^κ, but from now on, we focus rather on the wealth of this agent. We still denote by π_t^j the fraction of his total wealth to be invested at time $t \in [0,T]$ in the j-th asset, for $1 \le j \le N$. The vector $\pi_t = (\pi_t^1, \dots, \pi_t^N)^{\mathbf{T}}$ of size N is called the portfolio process. The portfolio process $\pi(.)$ is assumed to be $\mathbb{F}$-progressively measurable and such that $\sum_{j=1}^N \int_0^T (\pi_t^j)^2 dt < \infty$, a.s. The remaining fraction of wealth $\pi_t^0 = 1 - \sum_{i=1}^N \pi_t^i$ is invested in the riskless asset S^0. For an agent starting from a initial wealth x and taking a portfolio process $\pi(.)$, with the assumption of a self-financing portfolio, the dynamics of the wealth process would be:

$$dX_t^{x,\pi} = \sum_{i=0}^N \phi_t^i dS_t^i = X_t^{x,\pi} \sum_{i=0}^N \pi_t^i \frac{dS_t^i}{S_t^i}.$$

But here in addition to that, the agent is not only investing in the complete financial market $\mathcal{M}$, but also consuming a part of his wealth. A quantity to be taken into account is thus the consumption of the agent. In this utility maximization framework, the consumption of the agent is defined through the consumption rate $c(.)$. This means that during the time interval between date t and date $t + dt$, the agent is consuming a quantity $c_t dt$ of his wealth.

Definition 2.4.1 *A consumption process $c(.)$ is an $\mathbb{F}$-progressively measurable nonnegative process $c(.)$, satisfying*

$$\int_0^T c_s ds < +\infty, \ \ a.s.$$

Starting from an initial wealth x, the agent follows the investment strategy characterized by a portfolio $(\pi_t)_{t\ge0}$ and a consumption path $(c_t)_{t\ge0}$. The corresponding wealth process is then: $(X^{x,c,\pi})_{t\ge0}$. In the following, we consider only the set of positive wealth processes (see Definition 2.4.2).

We take the assumption of self-financing portfolios, from which we can substract a certain rate of consumption. That is, the dynamics of the wealth process $X^{x,c,\pi}$ starting from initial wealth x at time zero and taking the consumption path c_t and investment strategy π_t is:

$$dX_t^{x,c,\pi} = X_t^{x,c,\pi} \sum_{i=0}^N \pi_t^i \frac{dS_t^i}{S_t^i} - c_t dt. \tag{2.4.1}$$

We replace in (2.4.1) with the dynamics of the assets $S^i(.)$. The wealth process $X^{x,c,\pi}(.)$ is then the solution of the linear SDE:

$$dX_t^{x,c,\pi} = X_t^{x,c,\pi} \sum_{i=1}^{N} \pi_t^i \big(b_t^i dt + \sum_{j=1}^{N} \sigma_t^{ij} dW_t^j\big) + X_t^{x,c,\pi}\big(1 - \sum_{i=1}^{N} \pi_t^i\big) r_t dt - c_t dt \tag{2.4.2}$$

$$= X_t^{x,c,\pi}\big(r_t dt + \langle \pi_t, b_t - r_t \mathbf{1}_N\rangle dt + \langle \sigma_t^{\mathbf{T}} \pi_t, dW_t\rangle\big) - c_t dt, \tag{2.4.3}$$

with the initial condition $X_0^{x,c,\pi} = x > 0$, and where we recall that $\mathbf{1}_N$ is the vector of size N where each component is equal to 1.
In this equation we see the importance of the volatility vector:

$$\kappa_t := \sigma_t^{\mathbf{T}} \pi_t. \tag{2.4.4}$$

Thus throughout this work for sake of simplicity, we use the parameztrization by κ rather than π.

$$dX_t^{x,c,\kappa} = X_t^{x,c,\kappa}\big(r_t dt + \langle \kappa_t, \theta_t dt + dW_t\rangle\big) - c_t dt,\ X_0^{x,c,\kappa} = x.$$

Then we consider the discounted wealth process $\frac{X_t^{x,c,\kappa}}{S_t^0}$, where the discount factor S_t^0 is equal to $\exp(\int_0^t r_s ds)$. Applying Itô formula to $\frac{X_t^{x,c,\kappa}}{S_t^0}$ allows us to write the final expression of the discounted wealth process, that is, for all $0 \le t \le T$:

$$\frac{X_t^{x,c,\kappa}}{S_t^0} = x + \int_0^t \frac{X_s^{x,c,\kappa}}{S_s^0} \langle \kappa_s, \theta_s ds + dW_s\rangle - \int_0^t \frac{c_s}{S_s^0} ds. \tag{2.4.5}$$

2.4.2 The budget constraint

In the following we establish the budget constraint, with our set of hypotheses described previously. Let us notice first that the discounted wealth process $X_t^{x,c,\kappa}/S_t^0$ can be written in term of the Brownian motion $W^{\mathbb{Q}}$ and takes then a simple form:

$$\frac{X_t^{x,c,\kappa}}{S_t^0} = x - \int_0^t \frac{c_s}{S_s^0} ds + \int_0^t \frac{X_s^{x,c,\kappa}}{S_s^0} \langle \kappa_s, dW_s^{\mathbb{Q}}\rangle, \tag{2.4.6}$$

We recall that the state price density process H_t^0 (2.2.2) is defined by:

$$H_t^0 = \frac{Z_t^0}{S_t^0} = \exp\big(-\int_0^t r_s ds - \int_0^t \langle \theta_s, dW_s\rangle - \frac{1}{2}\int_0^t ||\theta_s||^2 ds\big).$$

Using Itô's lemma applied to the functional $H_t^0 X_t^{x,c,\kappa} = Z_t^0 \frac{X_t^{x,c,\kappa}}{S_t^0}$ and (2.4.6), we get:

$$H_t^0 X_t^{x,c,\kappa} + \int_0^t c_u H_u^0 du = x + \int_0^t H_u^0 X_u^{x,c,\kappa} \langle \kappa_u - \theta_u, dW_u \rangle, \qquad (2.4.7)$$

for all $0 \leq t \leq T$.

Definition 2.4.2 *Given an initial endowment $x \geq 0$, we say that a consumption and portfolio process pair (c, π) is admissible at x and write $(c, \pi) \in \mathcal{A}(x)$, if (c, π) are $\mathbb{F}$-progressively measurable processes and the wealth remains nonnegative at all times, that is, if the wealth process $X^{x,c,\kappa}(.)$ corresponding to x, c, π, with $\kappa = \sigma^{\boldsymbol{T}} \pi$ satisfies:*

$$X_t^{x,c,\kappa} \geq 0, \text{ for } 0 \leq t \leq T, d\mathbb{P} \text{ almost surely.}$$

For $x < 0$, we set $\mathcal{A}(x) = \emptyset$.

For any $(c, \pi) \in \mathcal{A}(x)$, the process $H_t^0 X_t^{x,c,\kappa} + \int_0^t c_t H_t^0 dt$ on the left hand side of (2.4.7) is a continuous and nonnegative local martingale, hence it is a supermartingale. Thus by the properties of supermartingales, the following inequality holds:

$$\mathbb{E}\left[\int_0^T c_t H_t^0 dt + X_T^{x,c,\kappa} H_T^0\right] \leq x. \qquad (2.4.8)$$

This is the budget constraint, showing the fact that the expected discounted terminal wealth plus the expected discounted total consumption does not exceed the initial endowment.

Now we provide some alternative expressions for this budget constraint. Since the agent's wealth remains nonnegative for all $t \in [0, T]$, the following inequality also holds. This is the budget constraint written for the consumption only.

$$\mathbb{E}\left[\int_0^T c_t H_t^0 dt\right] \leq \mathbb{E}\left[\int_0^T c_t H_t^0 dt + X_T^{x,c,\kappa} H_T^0\right] \leq x. \qquad (2.4.9)$$

Also, using (2.4.6), we can write a similar budget constraint under the risk neutral probability:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{c_u}{S_u^0} du + \frac{X_T^{x,c,\kappa}}{S_T^0}\right] \leq x.$$

We remind that in the previous section we have seen the existing link between the state price density process H_t^0 and the GOP. Because the state density process appears in the previous expression, it is also possible to rewrite this expression in terms of the GOP. Here this budget constraint is expressed in terms of the GOP:

$$\mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt + \frac{X_T^{x,c,\kappa}}{G_T^*}\right] \leq x.$$

Along the same lines, we remind that the **benchmarked wealth process** is the wealth process expressed in units of the GOP. Using the properties of the GOP, it is then possible to write the dynamics of the benchmarked wealth process.

Proposition 2.4.1 *The dynamics of the benchmarked wealth process (the wealth process expressed in units of the Growth Optimal Portfolio) is, for any $t \in [0, T]$:*

$$d\Big(\frac{X_t^{x,c,\kappa}}{G_t^*}\Big) = -\frac{c_t}{G_t^*}dt + \frac{X_t^{x,c,\kappa}}{G_t^*}\langle \kappa_t - \theta_t, dW_t\rangle,$$

Proof. This comes from the dynamics of $X_t^{x,c,\kappa}H_t^0$ and the relation between the GOP and the state density price process H_t^0. ■

2.4.2.1 The importance of the wealth process

Thus we see that the wealth plays a very important role in the usual utility maximization framework, such as it is developed by [KS98] and other authors. The starting point of the problem is an agent with a fixed initial wealth x. Then, the budget constraint is expressed in terms of the wealth of the agent. And the expression of the wealth $X_t^{x,c,\kappa}$ at time t is expressed in a convenient way in terms of the other parameter of the problem.
In the classical framework, the expected utility maximization problem is parametrized by wealth.
On the contrary, in the framework adopted by the economists, the expected utility maximization problem is parametrized by consumption.
In Chapter 4, we see how it is possible to unify these two approaches.

2.4.3 Utility functions

The representative agent taking actions in the complete market $\mathcal{M}$ seeks to maximize his expected utility, which describes his aversion to risk. Here we define utility functions and precise their properties.

Definition 2.4.3 *We call a concave, non decreasing, twice differentiable function* $U : [0, \infty[\to]0, \infty[$ *a utility function.*

Here we precise the properties of the utility functions that the agent wants to maximize.
The utility function U satisfies the Inada conditions:

$$\lim_{x \to 0} U'(x) = \infty$$

and:

$$\lim_{x \to +\infty} U'(x) = 0.$$

Due to all these assumptions, the inverse $I :]0, \infty[\to]0, \infty[$ of the function $U'(.)$ exists for every $t \in [0, T]$ and is continuous and strictly decreasing with $\lim_{x \to 0} I'(x) = \infty$ and $\lim_{x \to +\infty} I'(x) = 0$, also satisfying the Inada conditions.
In the following, we define the Fenchel transform of U.

Definition 2.4.4 *Let* U *be an utility function. The Fenchel transform* $\tilde{U}$ *of* U *is the function satisfying, for any* $y \in \mathbb{R}$*:*

$$\tilde{U}(y) = \sup_{x \in \mathbb{R}} \{\tilde{U}(x) - xy\},$$

For the properties of the function $\tilde{U}$ that will be useful in the following sections, we take a lemma from [KS98].

Lemma 2.4.1 *Let* U *be an utility function as previously defined. Let* $\tilde{U}$ *be the Fenchel tranform of* U*. Then* $\tilde{U}$ *is convex, nonincreasing, on* $]0, +\infty[$ *and satisfies, for all* $y > 0$*:*

$$\tilde{U}(y) = U(I(y)) - yI(y).$$

The function $\tilde{U}$ *is differentiable and its derivative* $\tilde{U}'$ *is defined, continuous, nondecreasing on* $]0, +\infty[$ *and, for all* $y > 0$*:*

$$\tilde{U}'(y) = -(U)^{-1}(y) = I(y).$$

Moreover, for all $x > 0$*,*

$$U(x) = \inf_{y > 0} \{\tilde{U}(y) + xy\}.$$

Also, for a fixed $x > 0$*, the function*

$$y \to \tilde{U}(y) + xy$$

is uniquely minimized over $\mathbb{R}^+$ *by* $y = U'(x)$ *that is, there is the relation:*

$$U(x) = \tilde{U}(U'(x)) + xU'(x).$$

Example 2.4.1 *We give here simple examples of Fenchel transform. For the logarithmic utility function,* $U(x) = \ln(x)$, $\tilde{U}(y) = -\ln(y) - 1$. *For the power utility function,* $U(x) = \frac{x^\alpha}{\alpha}$, $\tilde{U}(y) = \frac{y^{-q}}{q}$, $q = \frac{\alpha}{1-\alpha}$.

Definition 2.4.5 *A preference structure is a pair of functions* (U^1, U^2) *with* $U^1 : [0,T]\times]0,\infty[\to]0,+\infty[$ *and* $U^2 :]0,\infty[\to]0,+\infty[$ *such that for each* $t \in [0,T]$, $U^1(t,.)$ *is a utility function,* U^1_c *denotes differentiation with respect to the second argument:*

$$U^1_c(t,c) = \frac{\partial}{\partial x} U^1(t,c),$$

and the function $U^2(.)$ *is a utility function.*

From now on, we assume that the representative agent has the preference structure (U^1, U^2).
The function $U^1 : [0,T]\times]0,\infty[\to]0,+\infty[$ is called the agent's utility from consumption function. At each time t, it will give the utility $U^1(t, c_t)$ associated to a consumption rate c_t. We have the associated function $I_1 : [0,T]\times]0,\infty[\to]0,+\infty[$, and the Fenchel transform $\tilde{I}_1(t,y)$ defined as before.
The function $U^2 :]0,\infty[\to]0,+\infty[$ is called the utility from terminal wealth. We denote it $U^2(x)$. The associated definitions of I_2 and $\tilde{U}^2$ hold.

2.4.4 The maximization problem

The representative agent may maximize his utility from terminal wealth, or his utility from consumption over the planning horizon, or a combination of these two quantities. More precisely, his objective is to find an optimal pair (c^*, κ^*) for the problem of maximizing expected total utility over $[0,T]$. We recall that the volatility vector is denoted for all $t \in [0,T]$ by $\kappa_t := \sigma_t^{\mathbf{T}} \sigma_t$. Along the lines of [KS98], we introduce the following notations.

The representative agent, starting from an initial wealth may maximize his expected utility from consumption:

Problem 1

$$V_1(x) = \left\{ \sup_{(c,\kappa)\in\mathcal{A}_1(x)} \mathbb{E}\left[\int_0^T U^1(t,c_t)dt\right], X_T^{x,c,\kappa} = 0 \right\},$$

with $\mathcal{A}_1(x)$ *the set of admissible porfolios for the maximization from consumption problem of an agent starting from wealth* x:

$$\mathcal{A}_1(x) = \{(c,\kappa) \in \mathcal{A}(x); \mathbb{E}\int_0^T \min[0, U^1(t,c(t))]dt > -\infty\}$$

This can also be rewritten as:

$$\sup_{c\geq 0}\mathbb{E}\left[\int_0^T U^1(t,c_t)dt\right]$$

under the budget constraint:

$$\mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt\right]\leq x,$$

The representative agent may also maximize his expected utility from terminal wealth:

Problem 2

$$V_2(x)=\sup_{\kappa\in\mathcal{A}_2(x)}\mathbb{E}\left[U^2(X_T^{x,\kappa})\right],$$

under the budget constraint:

$$\mathbb{E}\left[\frac{X_T^{x,\kappa}}{G_T^*}\right]\leq x,$$

where $\mathcal{A}_2(x)$ is the set of admissible portfolios for the maximization from terminal wealth problem of an agent starting from wealth x:

$$\mathcal{A}_2(x)=\{\kappa\in\mathcal{A}(x);\mathbb{E}\min[0,U^2(X_T^{x,\kappa})]dt>-\infty\}$$

Throughout this work we mostly explore the last case, the maximization of expected utility from consumption and terminal wealth.

Problem 3

$$V_3(x)=\sup_{(c,\kappa)\in\mathcal{A}_3(x)}\mathbb{E}\left[\int_0^T U^1(t,c_t)dt+U^2(X_T^{x,c,\kappa})\right], \tag{2.4.10}$$

under the budget constraint:

$$\mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt+\frac{X_T^{x,c,\kappa}}{G_T^*}\right]\leq x,$$

where the set of admissible portfolios $\mathcal{A}_3(x)$ for this problem is:

$$\mathcal{A}_3(x)=\mathcal{A}_1(x)\cap\mathcal{A}_2(x).$$

The function $V_3(x)$ is called the value function of Problem 3. Similarly, we denote by $V_1(x)$ the value function for the problem of maximization of utility from consumption only (Problem 1), and $V_2(x)$ the value function for the problem of maximization of utility from terminal wealth only (Problem 2).

2.5 The optimal consumption

2.5.1 Expression of the optimal consumption path

In this part, we are still considering the complete market $\mathcal{M}$ previously defined. Still focusing on consumption, we give the expression of the optimal consumption path, solution of the maximization of utility from consumption and terminal wealth.

Theorem 2.5.1 *From [KS98]. For an agent with initial wealth x, taking a consumption $c(.)$ and porfolio $\kappa(.)$, and preference structure (U_1, U_2), consider the expected utility maximization problem from consumption and terminal wealth (Problem 3), that is:*

$$\sup_{(c,\kappa)\in\mathcal{A}_3(x)} E\left[\int_0^T U^1(t,c_t)dt + U^2(X_T^{x,c,\kappa})\right] \; s.c. \; \mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt + \frac{1}{G_T^*}X_T^{x,c,\kappa}\right] \leq x. \tag{2.5.1}$$

We call $x \to \mathcal{Y}_3(x)$ the inverse of the function:
$y \to \mathbb{E}\left[\int_0^T \frac{1}{G_t^*} I_1(t, \frac{y}{G_t^*})dt + \frac{1}{G_T^*} I_2(\frac{y}{G_T^*})\right]$. *Then the optimal consumption path is $c_t^* = I_1(t, \frac{\mathcal{Y}_3(x)}{G_t^*})$, for all $0 \leq t \leq T$. And the optimal terminal wealth is $X_T^{*,x} = I_2(\frac{\mathcal{Y}_3(x)}{G_t^*})$.*

Proof. Looking into the details of the classical theory from [KS98], here rewritten in terms of the GOP rather than the state price density process, we consider the optimization problem (Problem 3):

$$V_3(x) = \sup_{(c,\zeta_T)\in\mathcal{A}_3(x)} E\left[\int_0^T U^1(t,c_t)dt + U^2(\zeta_T)\right] \text{ s.c. } \mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt + \frac{1}{G_T^*}\zeta_T\right] \leq x, \tag{2.5.2}$$

where ζ_T is a $\mathcal{F}_T$-measurable random variable. Then, if $y > 0$ is a Lagrange multiplier that enforces this constraint, the problem reduces to the maximization of:

$$\mathbb{E}\left[\int_0^T U^1(t,c_t)dt + U^2(\zeta_T)\right] - y\mathbb{E}\left[\int_0^T \frac{c_t}{G_t^*}dt + \frac{\zeta_T}{G_T^*}\right]$$

Thus:

$$\mathbb{E}\left[\int_0^T \big(U^1(t,c_t) - y\frac{c_t}{G_t^*}\big)dt + \big(U^2(\zeta_T) - y\frac{\zeta_T}{G_T^*}\big)\right]$$
$$\leq \mathbb{E}\left[\int_0^T \tilde{U}^1(t,\frac{y}{G_t^*})dt + \tilde{U}^2(\frac{y}{G_T^*})\right],$$

with equality if and only if for any $0 \leq t \leq T$:

$$c_t^* = I_1(t,\frac{y}{G_t^*}), \text{ and } \zeta_T^* = I_2(\frac{y}{G_T^*}).$$

The first equation involving consumption comes from the previously mentioned properties of $\tilde{U}^1$. This is the candidate optimal consumption process. The second equation is the candidate optimal wealth process.
We define the following functions. For the problem of maximizing the utility from consumption only, we define the function:

$$\mathcal{X}_1 : y \to \mathbb{E}\left[\int_0^T \frac{1}{G_t^*} I_1(t,\frac{y}{G_t^*})dt\right]. \tag{2.5.3}$$

And for the problem of maximizing utility from terminal wealth we define:

$$\mathcal{X}_2 : y \to \mathbb{E}\left[\frac{1}{G_T^*} I_2(\frac{y}{G_T^*})\right]. \tag{2.5.4}$$

For the problem of maximizing the utility from consumption and terminal wealth, we define the function:

$$\mathcal{X}_3 : y \to \mathbb{E}\left[\int_0^T \frac{1}{G_t^*} I_1(t,\frac{y}{G_t^*})dt + \frac{1}{G_T^*} I_2(\frac{y}{G_T^*})\right]. \tag{2.5.5}$$

We can check that for any $y \in \mathbb{R}$: $\mathcal{X}_1(y) + \mathcal{X}_2(y) = \mathcal{X}_3(y)$.
The function $\mathcal{X}_3$ is nonincreasing and it has a strictly decreasing inverse.
We define the function:

$$\mathcal{Y}_3 : x \in \mathbb{R} \to \mathcal{Y}_3(x),$$

such that $\mathcal{Y}_3(.)$ is the inverse of $\mathcal{X}_3(.)$. This implies that $\mathcal{X}_3(\mathcal{Y}_3(x)) = x$. Similarly we define $\mathcal{Y}_1$ the inverse of $\mathcal{X}_1$ and $\mathcal{Y}_2$ the inverse of $\mathcal{X}_2$.

At the optimum the budget constraint is saturated. The only value of $y > 0$ for which the budget constraint is satisfied with equality is such that:

$$\mathbb{E}\left[\int_0^T \frac{1}{G_t^*} I_1(t,\frac{y}{G_t^*}) + \frac{1}{G_T^*} I_2(\frac{y}{G_T^*})\right] = x,$$

that is $\mathcal{X}_3(y) = x$ or equivalently, the value of the Lagrange multiplier is:

$$y = \mathcal{Y}_3(x).$$

This gives the optimal consumption process:

$$c_t^* = I_1(t, \frac{\mathcal{Y}_3(x)}{G_t^*}). \tag{2.5.6}$$

■

For an agent who wants to consume as much as he can, the optimal consumption process gives him the maximum that the agent can consume, when taking into accounts the constraint given by the budget constraint.
Taking this equality (2.5.6) at $t = 0$, the initial optimal consumption satisfies:

$$c_0^* = I_1(0, \mathcal{Y}_3(x)). \tag{2.5.7}$$

What we put in evidence here is that there is a relation between the optimal initial consumption and the Lagrange multiplier, which can be seen even better when it is written as:

$$U_c^1(0, c_0^*) = \mathcal{Y}_3(x).$$

As we have seen in Chapter 1, in Economics the focus is on consumption. It is therefore not unexpected that the initial consumption appears, in particular in the Ramsey Rule.
On the contrary, in the utility maximization framework usually developped in finance ([KS98]), the focus is more on the wealth of the representative agent.
The quantity c_t is a rate of consumption at time t. Consumption always appears through the integral of this consumption rate, that is: $\int_0^t c_s ds$, but the initial consumption c_0 does not usually appear on its own.

The previous expression (2.5.7) shows a link between initial wealth x and initial optimal consumption c_0^*. This gives us a first link between the economic and the financial approach of our problem. We explore this in the following paragraph.

2.5.2 Link between initial consumption and initial wealth

In this part we explore the link between initial wealth and optimal consumption. It is intuitively easy to understand what the initial wealth, or initial

endowment represents: it is the wealth that the agent possesses when he enters in a consumption and investment strategy. But it is more difficult to understand what the "initial consumption" represents.
Here we express the non linear relation existing between the initial wealth and the initial optimal consumption c_0^*. For this purpose, in this paragraph, we consider the problem from maximization of the expectation from consumption only (Problem 1). On one hand, we know that the optimal consumption is given by:

$$c_0^* = I_1(0, \mathcal{Y}_1(x)).$$

On the other hand, using the definition of $\mathcal{Y}_1(x)$, we get:

$$x = \mathcal{X}_1(\mathcal{Y}_1(x)) = \mathcal{X}_1(U_c^1(0, c_0^*)) = \mathbb{E}\left[\int_0^T I_1(s, \frac{U_c^1(0, c_0^*)}{G_s^*})\frac{1}{G_s^*}ds\right].$$

This is an expression of the initial wealth as a function of the initial optimal consumption. Thus, there is a bijection between x the initial wealth and c_0^* the initial consumption.
In the usual framework, the initial wealth of the representative agent is given and the initial optimal consumption is deduced from the optimization. But instead, we could say that the initial optimal consumption is given and deduce the initial optimal wealth.
In this work, we develop further the idea of parametrizing expected utility maximization problem through consumption instead of wealth, especially in Section 2.7.2 and Chapters 4 and 5.

2.6 Optimal consumption and wealth: examples and consequences

2.6.1 The GOP: the optimal porfolio for logarithmic utility

According to Platen [PH06] (and Definition 2 of this work), one way to define the GOP is to use the fact that it is the portfolio which maximizes the expected logarithmic utility from terminal wealth. This is not the definition that we have used previously to derive the dynamics of the GOP.
But here, in the complete market $\mathcal{M}$ previously defined, and considering the maximization problem between 0 and T, we show how we can find again the dynamics of the GOP with this second definition, and this expression is coherent with the expression found before.

If we choose a preference structure for the agent where $U^1 \equiv 0$ and $U^2(x) = \log(x)$, the maximization problem (2.4.10) becomes Problem 2:

$$\sup_{\kappa} \mathbb{E}[\ln(X_T^{x,\kappa})] \text{ s.c. } \mathbb{E}[\frac{X_T^{x,\kappa}}{G_T^*}] \leq x.$$

This is the maximization of utility from terminal wealth, with a logarithmic utility.
For an agent starting at time 0 from a wealth x, the solution to the maximization problem gives also the optimal terminal wealth X_T^*:

$$U_x^2(X_T^*) = \mathcal{Y}_3(x)\frac{1}{G_T^*},$$

which can also be written:

$$U_x^2(X_T^*) = U_x^2(x)\frac{1}{G_T^*}. \tag{2.6.1}$$

With our choice of preference structure, the optimal terminal wealth is thus:

$$X_T^{*,x} = x \exp\left(\int_0^t r_s ds + \int_0^t \langle\theta_s, dW_s\rangle + \frac{1}{2}\int_0^t ||\theta_s||^2 ds\right).$$

We recognize here the expression of the GOP. This proves that the GOP is the optimal portfolio for the logarithmic utility. This is how the dynamics of the GOP is derived in Definition 2.

For the sake of the illustration, we provide some more examples. If $U^2(x) = \frac{x^{1-\alpha}}{1-\alpha}$, we obtain:

$$X_T^{*,x} = x\left(\frac{S_T^\theta}{S_0^\theta}\right)^{\frac{1}{\alpha}} = x(G_T^*)^{1/\alpha}.$$

This means that there is a simple relation between the optimal wealth $X^{*,x}(.)$ of a agent with initial wealth x and the optimal wealth $X^{*,1}(.)$ of an agent with initial wealth 1.

$$X_T^{*,x} = xX_T^{*,1}.$$

For more general utility function U^2 (and the associated function I_2, such that $I_2 = (U_x^2)^{-1}$, we still have a relation between the optimal terminal wealth and the GOP:

$$X_T^{*,x} = I_2\left(\frac{x}{G_T^*}\right).$$

2.6.2 Link between GOP and marginal utility from consumption

In this part, we are still considering a utility maximization problem in a complete market, between dates 0 and T.
We recall the equation involving the marginal utility from optimal consumption:

$$U_c^1(t, c_t^*) = \frac{U_c^1(0, c_0^*)}{G_t^*}. \tag{2.6.2}$$

This equation gives a relation between the GOP and the **marginal utility from consumption**. The important thing is that this result holds pathwise.

In the next section (2.7.2), by taking the expectation of the left and right hand side of equation (2.6.2), we obtain the Ramsey Rule (equation (1.2.1)). We already see that we are getting closer to this formula: what we have obtained so far already involves the marginal utility from consumption. However the result in (2.6.2), expressing the marginal utility from optimal consumption holds pathwise, not only in expectation, therefore it is more precise.
And in addition to that, we have here a link between the GOP and the marginal utility from consumption, which is one of the quantities that will be of great interest for us.
Example. A very common example of utility from consumption function $U^1(t, c_t)$ is a separable utility function. More precisely, let $\beta > 0$ be the pure time preference parameter and a function $u : \mathbb{R} \to \mathbb{R}^+$ three times differentiable. The utility from consumption function is:

$$U^1(t, x) = \exp(-\beta t)u(x).$$

The parameter β is the pure time preference parameter that usually appears in economics. Then we get the initial condition: $\mathcal{Y}_3(x) = u'(c_0^*)$. On the other hand, replacing in (2.6.2) with this choice of utility function U^1, we find the relation:

$$\exp(-\beta t)u'(c_t^*) = u'(c_0^*)\frac{1}{G_t^*}. \tag{2.6.3}$$

With the logarithmic utility $u(x) = \log(x)$, equation (2.6.3) becomes:

$$c_t^* = c_0^* e^{-\beta t} G_t^*.$$

Hence the optimal consumption is proportional to the GOP. This would show the ability of the agent to choose well between risky or not risky assets, and to choose an optimal consumption that ressembles the best performing

portfolio.
■

Remark 2.6.1 *The process:*

$$t \to e^{-\beta t} \frac{u'(c_t^*)}{u'(c_0^*)},$$

is a supermartingale. Hence for any $0 \leq s \leq t \leq T$*:*

$$e^{-\beta(t-s)} \mathbb{E}[u'(c_t^*)|\mathcal{F}_s] \leq u'(c_s^*).$$

In the work of Hansen and Scheinkman [HS09], the process $t \to e^{-\beta t} \frac{u'(c_t^*)}{u'(c_0^*)}$ *is called a Stochastic Discount Factor.*
We underline however the fact that this process is linked to the state price density process H_t^0*, (hence it is not a discount factor in the sense of* S_t^0*).*

Thus, we have seen in this section the link between the GOP and the optimal consumption path and the fact that this optimal consumption path is very closely related to the GOP.

2.6.3 Towards a dynamic point of view

Our goal is a dynamic programming point of view, and we would like to write the maximization problem not only between initial date 0 and horizon T but also between any dates t and T such that $0 \leq t \leq T$.
On the other hand, since consumption plays an crucial role in our problem, a question arises: why do we need to consider the expected utility maximization problem from consumption AND terminal wealth? Why not the expected utility maximization problem from consumption ONLY?
In order to answer this question, consider that the terminal wealth is $X_T^{x,c,\kappa} = 0$. But at an intermediate date $0 \leq t \leq T$, the wealth is not equal to zero.
Thus, even if between 0 and T the problem reduces to a expected utility maximization from consumption only (Problem 1), this is no longer the case between dates t and T. If we want to adopt a more dynamic point of view, between dates t and T, the wealth is not equal to zero, we have to take it into account and to maximize the expected utility from consumption AND terminal wealth.

Consequently, we use the maximization of utility from consumption and terminal wealth instead of the maximization of utility from consumption only. For all dates s such that $t \leq s \leq T$, we define the state price density process:

$$\begin{aligned} H^0_{t,s} &= \exp\left(-\int_s^t r_u du - \int_s^t \langle \theta_u, dW_u \rangle - \frac{1}{2}\int_s^t ||\theta_u||^2\right) \\ &= \frac{H^0_s}{H^0_t}. \end{aligned}$$

Consider now an intermediate date $0 \leq t \leq T$, then the utility maximization problem between t et T, starting from a wealth $X^{c,\kappa}_t$ at time t, is written

$$V(t, X^{c,\kappa}_t) = esssup_{(c,\pi)}\mathbb{E}\left[\int_t^T U^1(s, c_s)ds + U^2(T, X^{c,\kappa}_T)|\mathcal{F}_t\right],\ a.s.,$$

under the budget constraint:

$$\mathbb{E}\left[\int_t^T c_s H^0_{t,s} ds + H^0_{t,T} X^{c,\kappa}_T|\mathcal{F}_t\right] \leq X^{*,c,\kappa}_t \text{ a.s.}$$

If the optimum exists, one has:

$$V(t, X^{*,x}_t) = \mathbb{E}\left[\int_t^T U^1(s, c^*_s)ds + U^2(T, X^{*,x}_T)|\mathcal{F}_t\right]. \tag{2.6.4}$$

One can check that the optimal processes c^*, κ^*, and $X^{*,x}$ defined in the previous sections are also solutions of this conditional problem. Instead of starting from an initial wealth x at time 0, one starts from a wealth $X^{*,x}_t$ at time t.
For our purpose, we consider the maximization problem between 0 and T. But the methodology is the same for the maximization problem between t and T starting from a wealth $X^{*,x}_t$ at time t.

2.7 The Ramsey Rule

2.7.1 Interest rates: definitions

First of all, we need to recall definitions relative to interest rates, taken from [BF01], [MR00], [ElK], [HJM98].
Let T^H be a time horizon. Throughout this work, by zero-coupon bond of maturity T we mean a financial security paying to its holder one unit of cash at a prespecified date $T \leq T^H$ in the future. The price of a zero-coupon bond

of maturity T at any instant $0 \leq t \leq T$ will be denoted by $B(t,T)$. We will usually assume that, for any fixed maturity T, the bond price $(B(t,T))_{0\leq t\leq T}$ is a strictly positive and adapted process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ previously defined.

Definition 2.7.1 *An adapted process $Y_t(T)$, defined by the formula:*

$$Y_t(T) = -\frac{1}{T-t}\ln B(t,T),$$

for all $0 \leq t \leq T \leq T^H$ is called the yield to maturity on a zero-coupon bond with maturity T. On the other hand, given a yield to maturity process $Y_t(T)$, the corresponding zero-coupon price is given by:

$$B(t,T) = e^{-Y_t(T)(T-t)}.$$

The term structure of interest rates, or yield curve is the function that relates the yield $Y_t(T)$ to maturity T: $T \to Y_t(T)$.

We give here another definition concerning interest rates, which is useful in the following chapters.

Definition 2.7.2 *The time t forward rate $f(t,T)$ for the maturity date $T \leq T^H$ is defined as:*

$$f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T).$$

On the other hand, if a family $f(t,T)_{t\leq T}$ is specified, as in [HJM98], then given such a family $f(t,T)$ the bond prices are defined by setting, for all $0 \leq t \leq T$:

$$B(t,T) = exp\left(-\int_t^T f(t,u)du\right).$$

2.7.2 The Ramsey Rule

In this part we show how we can deduce the Ramsey Rule (1.2.1) from the expression of the optimal consumption path (2.6.3), in the case of a utility from consumption function taking the form: $U^1(t,x) = e^{-\beta t}u(x)$.

Theorem 2.7.1 *In the complete market $\mathcal{M}$, with a choice of utility function $U^1(t,x) = e^{-\beta t}u(x)$, the Ramsey Rule links the yield curve and the optimal consumption path.*

$$R_0(T) = \beta - \frac{1}{T}\ln\left(\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c^*_T)}{u'(c^*_0)}\right]\right), \tag{2.7.1}$$

Proof. Starting from equation (2.6.3) and taking the expectation on both sides we obtain:

$$\exp(-\beta T)\frac{u'(c_T^*)}{u'(c_0^*)} = \exp\left(-\int_0^T r_s ds\right)\exp\left(-\int_0^T \langle\theta_s, dW_s\rangle - \frac{1}{2}\int_0^T ||\theta_s||^2 ds\right)$$

$$\exp(-\beta T)\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right] = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_0^T r_s ds\right)\exp\left(-\int_0^T \langle\theta_s, dW_s\rangle - \frac{1}{2}\int_0^T ||\theta_s||^2 ds\right)\right].$$

This last expression can also be written:

$$\exp(-\beta T)\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right] = \mathbb{E}^{\mathbb{P}}\left[\exp(-\int_0^T r_s ds)Z_T^0\right],$$

where we recognize the change of probability density between the historical probability measures $\mathbb{P}$, and the risk-neutral probability measure $\mathbb{Q}$. Hence we get:

$$\exp(-\beta T)\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right)\right].$$

This last expression is a relation between the expectation of the marginal utility from optimal consumption taken under the historical probability $\mathbb{P}$ and a second term, where the expectation is taken under the pricing probability $\mathbb{Q}$. In the complete market $\mathcal{M}$ in which the agent acts in this part, the second term of the previous equation is the zero-coupon price of a bond of maturity T. We denote by $B(0,T)$ this zero-coupon price, such that:

$$B(0,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right)\right].$$

This gives finally:

$$B(0,T) = \exp(-\beta T)\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right]. \tag{2.7.2}$$

Thus we have a relation linking the zero-coupon price and the expectation of the marginal utility from optimal consumption under the probability $\mathbb{P}$. The reason why we have made appear these zero-coupon prices was in order to introduce the presence of interest rates in the previous equation. We remind of the fact that the relation between zero-coupon price $B(0,T)$ and $R_0(T)$, is:

$$R_0(T) = -\frac{1}{T}\log(B(0,T)). \tag{2.7.3}$$

And then equation (2.7.2) becomes for all $t \in [0,T]$:

$$R_0(T) = \beta - \frac{1}{T}\ln\left(\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right]\right),$$

that is the equation (1.2.1), also called the Ramsey rule. ∎

Remark 2.7.1 *Similarly, equation (2.7.1) holds for all* $0 \leq t \leq T$*:*

$$Y_t(T) = \beta - \frac{1}{T-t} \ln \left(\mathbb{E}^{\mathbb{P}} \left[\frac{u'(c_T^*)}{u'(c_t^*)} \middle| \mathcal{F}_t \right] \right).$$

Remark 2.7.2 *Similar results (1.2.1) can be obtained in the infinite horizon case, if we assume that the consumption* $c(.)$ *and portfolio* $\kappa(.)$ *processes satisfy the assumptions from [KS98].*

Thus, in the framework of the complete financial market $\mathcal{M}$ described previously we have been able to link the yield curve with the marginal utility from consumption.
This is a formula commonly used by economists ([Gol02], for example). In this expression, the parameter β is the pure time preference parameter. Chosing a good value for this parameter is an important issue, see for instance the work of Weitzman [Wei98].
The expression of the Ramsey Rule shows that the interest rate curve $R_0(T)$ depends very strongly on the optimal consumption. We can also remark once more that in the Ramsey Rule, the initial consumption c_0^* plays an important role.
It is finally crucial to notice that this formula has been established in the case of the existence of a complete financial market (the financial market $\mathcal{M}$ previously described).
In finance, the spot rate r_t is considered as an intrinsic data of the economy. In particular, in a complete market, with a market model it is possible to deduce the zero coupons $B(0,t)$ and the yield curve $R_0(T)$.
Thus, in a complete market, the left hand side of the Ramsey Rule (1.2.1) is given by the market. It does not depend on the problem we consider or on the wealth invested, or on the chosen utility functions.

Equation (2.6.3) shows that the optimal consumption is a certain function of the GOP. This leads us to the following remark: we have started with a general consumption process $c(.)$ with few assumption on its dynamics. But using equation (2.6.2), we have a more precise idea of the dynamics of the optimal consumption process.
We do not know much about the consumption process c_t. However, we know that the optimal consumption process c_t^* is a function of H_t^0. The class of processes that can be optimal is relatively restricted, because these processes have to satisfy certain properties. In particular $U_c^1(t, c_t^*)S_t^\theta$ has to be a constant.

In the financial framework we have started from a process c_t with very few hypothesis and the optimal consumption framework c_t^* has been deduced from the solution of the utility maximization problem.

Instead, in several references in economic literature the consumption is considered as given. For example Gollier ([Gol]) makes an a priori assumption on the form of the optimal consumption:

$$c_t^* = c_0^* \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma dW_t\right)$$

Another example is found in Scheinkman et al. [HS09] and Breeden's consumption based asset pricing model ([Bre89]). In this case, the consumption process at the equilibrium is assumed to have the following dynamics:

$$c_t^* = c_0^* \exp\left(\int_0^t X_s^0 ds + \int_0^t \sqrt{X_s^f} \nu_f dW_t^f + \nu_0 W_t^0\right),$$

where $(X^0)_{t\geq 0}$ is an Ornstein-Uhlenbeck process and $(X^f)_{t\geq 0}$ is a square root process, its dynamics is a solution of $dX_t^f = \xi_f(\bar{x}^f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dW_t^f$.

2.8 Pricing with the GOP: The Benchmark Approach

The benchmark approach, described in [PH06], assumes in a general setting the existence of a tradable numeraire, or benchmark that is such that all nonnegative price processes, when expressed in units of benchmark are supermartingales. In the following we always use the GOP as a benchmark porfolio. Here we give a definition of a fair price in the sense of Platen.

Definition 2.8.1 *In a complete market, the fair price at time t for a pay-off Φ_T at time T is given by the benchmark pricing formula:*

$$P_t = S_t^\theta \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_T}{S_T^\theta}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_T}{G_{t,T}^*}|\mathcal{F}_t\right].$$

Other definitions of a fair price exist, for example the one given by Davis [Dav98].
In a complete market, in the usual pricing under the risk-neutral probability framework, discounted portfolios are $\mathbb{Q}$-martingales.
Here instead, benchmarked portfolios (i.e. discounted by the GOP) are

$\mathbb{P}$-martingales. Nevertheless this framework allows for pricing. This is the Benchmark approach.

Remark 2.8.1 *However, if an equivalent risk-neutral measure exists, fair prices coincide with risk neutral prices. Indeed, we have, for any* $0 \leq t \leq T$ *the following relation between risk-neutral pricing and pricing under the historical probability in the benchmark framework:*

$$P_t = \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_T}{G^*_{t,T}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\Phi_T}{G^*_{t,T}}\frac{Z^0_T}{Z^0_t}|\mathcal{F}_t\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi_T}{G^*_{t,T}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\Phi_T|\mathcal{F}_t\right].$$

This is the well-known risk-neutral pricing formula. In particular, for zero-coupon bonds, we have, for all $0 \leq t \leq T$*:*

$$B(t,T) = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{G^*_{t,T}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[exp(-\int_t^T r_s ds)|\mathcal{F}_t\right]$$

Thus it is possible to price under the risk-neutral probability or under the historical probability (with the GOP as a numeraire).

Using the fact that benchmarked prices are local martingales, it is possible to prove the well-known formula giving the dyamics of zero-coupon bond prices.

Proposition 2.8.1 *The dynamics of zeros-coupon bonds is, for all* $0 \leq t \leq T$*:*

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \langle \Gamma(t,T), dW_t + \theta_t dt \rangle,$$

where we remind that $W(.)$ *is the Brownian motion under the historical probability.*

Proof. The price of a benchmarked zero-coupon (that is discounted by the GOP), $B(t,T)/G^*_{t,T}$ is a $\mathbb{P}$ local martingale. Thus we choose to write its dynamics as:

$$d\left(\frac{B(t,T)}{G^*_{t,T}}\right) = \left(\frac{B(t,T)}{G^*_{t,T}}\right)\langle \Gamma(t,T) - \theta_t, dW_t \rangle. \tag{2.8.1}$$

In this last expression, $\Gamma(t,T)(.)$ is a vector of size N with terms $\Gamma^k(t,T)$, for all $k \in 1,\dots,N$ such that:

$$\langle \Gamma(t,T), dW_t \rangle = \sum_{k=1}^{N} \Gamma^k(t,T) dW_t^k.$$

Thus:

$$\frac{B(t,T)}{G^*_{t,T}} = B(0,T)\exp\left(\int_0^t \langle \Gamma(s,T) - \theta_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\Gamma(s,T) - \theta_s||^2 ds\right).$$

Then, by (2.8.1), the definition of a benchmarked portfolio (Definition 2.3.4), the dynamics of the GOP (2.3.5) and the application of the Itô formula we obtain the following SDE for the dynamics of the zero-coupon bond:

$$\begin{aligned}
&dB(t,T)\\
&= d(\frac{B(t,T)}{G^*_{t,T}})G^*_{t,T} + dG^*_{t,T}\frac{B(t,T)}{G^*_{t,T}} + d\langle \frac{B(t,T)}{G^*_{t,T}}, G^*_{t,T}\rangle.\\
&= B(t,T)(\langle \Gamma(t,T) - \theta_t, dW_t \rangle + r_t dt + ||\theta_t||^2 dt + \langle \theta_t, dW_t \rangle + \langle \Gamma(t,T) - \theta_t, \theta_t dt \rangle).
\end{aligned}$$

And finally we obtain:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \langle \Gamma(t,T), dW_t + \theta_t dt \rangle,$$

An analoguous expression can be found in [PH06]. Here we stress the fact that this expression has been obtained with $W(.)$ a Brownian motion under the historical probability. This is also consistent with modelling of zero-coupon bond dynamics under the risk-neutral probability, that is:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \langle \Gamma^{\mathbb{Q}}(t,T), dW_t^{\mathbb{Q}} \rangle,$$

■

Further remarks concerning the dynamics of zero-coupon bonds will be given in Chapters 3 and 5.

Until now in this work, we have assumed the existence of an equivalent risk-neutral measure. We have used the GOP as a numeraire and employed the historical probability as a pricing measure. This is the kind of pricing that has also been used in [Lon90], [BBP97], [Bec01].
This is very close to the work of [PH06] and we use many of his results. The primary difference is that the benchmark approach of [PH06] does not assume

the existence of an equivalent risk neutral probability measure, thus it is more general. Thus, the advantage of the benchmark approach is that it is disconnected from the assumption of the existence of an equivalent risk-neutral probability measure. It is still possible to price under the benchmark approach and thus to find information concerning the dynamics of zero-coupon bonds. The expressions of zero-coupon bond prices or of the yield curve hold for a finite horizon T. We have chosen $T \leq T^H$, we choose a finite horizon. For the moment, in this work we have not considered the case where $T \to +\infty$. In Economics, this case is often mentioned. But in finance, the density of the risk-neutral probability tends to zero when $T \to +\infty$, see for instance [Mar08, PH06]. This is a reason to use a pricing method under the historical probability, such as the benchmark method. For $T \to +\infty$ the GOP plays a special role. Indeed it outperforms the long term growth rate. For any strictly positive portfolio with value S_t^κ at time t, its long term growth rate $\tilde{g}^\kappa$ is the almost sure upper limit:

$$\tilde{g}^\kappa = \limsup_{T \to \infty} \frac{1}{T} \ln \left(\frac{S_T^\kappa}{S_0^\kappa} \right),$$

assuming that this limit exists for the GOP. This pathwise quantity does not involve any expectation. Then a theorem in [PH06] states that in a continuous financial market, the GOP G_t^* attains almost surely the greatest long term growth rate $\tilde{g}^\theta$ compared with all other strictly positive portfolios $S^\kappa(.)$, that is:

$$\tilde{g}^\theta \geq \tilde{g}^\kappa.$$

But throughout this work, we remain in the case of a long term but finite horizon.

2.9 A model for the GOP

In the remaining part of this Chapter we present results from [PH06] showing that a model approximating the GOP can be implied from historical data. Then, if we believe in this model for the GOP, in the framework of the benchmark approach it is possible to deduce a historical yield curve $R_t^{GOP}(T)$ and the dynamics of historical zero coupon bond prices $B^{GOP}(t,T)$ (without the assumption of the existence of a risk-neutral probability).

It is useful to be able to identify the GOP in practical terms, at least approximately, in order to be able to implement the benchmark approach. In order to determine the optimal fractions invested in each of the assets, one needs an accurate model and accurate estimates of the volatility and of the risk premium. However, an alternative route is to find proxies for the GOP. Then if we have such an approximation of the GOP, it is possible to deduce results concerning the dynamics of zero-coupons and the yield curve from this approximation of the GOP.

2.9.1 Approximation of the GOP

In Platen [PH06], [Pla04b], it is shown that under some conditions, a well diversified portfolio approximates the GOP, without any major assumption on the model. It is possible to put in evidence several world stock indices that provide a good approximation of the GOP. Hence it is possible to construct a proxy for the GOP from observable data.
In particular it is shown in [Pla04c] that the following examples are good approximations of the GOP.

Example 2.9.1 *The Equi-Value Weighted index is a portfolio for which, at each date t, equal fractions of its total value are invested in each of the assets. Thus, the fractions invested in each asset are readjusted at each date, so that the total value of the portfolio is equally distributed between the assets.*

Example 2.9.2 *An accumulation index is a portfolio which holds one unit of each asset for the entire time period.*

2.9.2 GOP under the MMM (Minimal Market Model)

Here we present another model from [PH06], which is an approximation of the GOP.
It is possible to derive an alternative model for the long term dynamics of the GOP from economic arguments. The discounted GOP drift, which models the long term trend of the economy is chosen as the key parameter process. This leads to the Minimial Market Model or MMM, described by Platen and Heath [PH06], where the discounted GOP is a time transformed squared Bessel of dimension 4.
Introducing $\tilde{W}_t$, the standard Brownian motion characterized by the SDE:

$$d\tilde{W}_t = \frac{1}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k dW_t^k.$$

Then the SDE of the discounted GOP becomes:

$$d\bar{S}_t^\theta = \bar{S}_t^\theta |\theta_t| (|\theta_t| dt + d\tilde{W}_t). \tag{2.9.1}$$

The discounted GOP drift models the long term trend of the economy. This drift is an important link with the long term growth of the economy and it is chosen to be the key parameter $\alpha_t^\theta := \bar{S}_t^\theta |\theta_t|^2$, also called the Net Market Trend. Then the SDE for the discounted GOP becomes:

$$d\bar{S}_t^\theta = \alpha_t^\theta dt + \sqrt{\bar{S}_t^\theta \alpha_t^\theta} d\tilde{W}_t.$$

where the square root of the discounted GOP appears in the diffusion coefficient. In this framework, the discounted GOP follows the dynamics of a time transformed squared Bessel process of dimension four. To make this appear, Platen et al. introduce the GOP time: $\varphi_t = \frac{1}{4}\int_0^t \alpha_s^\theta ds$. The discounted GOP process written in GOP time is such that:

$$X(\varphi_t) = \bar{S}_t^\theta.$$

Its SDE becomes:

$$dX(\varphi_t) = 4d\varphi_t + \sqrt{4X(\varphi_t)} d\tilde{W}_{\varphi_t}, \text{ with } d\tilde{W}_{\varphi_t} = \sqrt{\frac{\alpha_t^\theta}{4}} d\tilde{W}_t, \text{ and } X_0 = \bar{S}_0^\theta. \tag{2.9.2}$$

Because of (2.9.2), X is a squared Bessel proces of dimension 4. Therefore the process $\bar{S}^\theta(.)$ is a time transformed squared Bessel process of dimension four. Concerning the so-called GOP time φ_t, it is directly observable in the market. Indeed, it is equal to $\varphi_t = \langle \sqrt{\bar{S}^\theta} \rangle_t$. Hence the net market trend α_t^θ is given by:

$$\alpha_t^\theta = 4\frac{d\varphi_t}{dt} = 4\frac{d\langle \sqrt{\bar{S}^\theta} \rangle_t}{dt}.$$

The net market trend α_t^θ is an observable financial quantity. It measures the market activity. Furthermore, in [PH06], there is the following assumption that the discounted GOP drift is an exponentially growing function of time: $\alpha_t^\theta = \alpha_0 \exp(\eta t)$, with the nonnegative initial value $\alpha_0 > 0$, which depends on the initial date and initial value of discounted GOP and $\eta > 0$ is called the net growth rate of the market. This assumption comes from the fact that the discounted index they are trying to model seems, on average, to grow exponentially. For this model of the discounted GOP one needs to specify the initial values $\bar{S}_0^\theta$ and α_0 and the net growth rate η. According to [PH06] this is a realistic framework to price derivatives (in the long term also).

2.9.3 The yield curve deduced from the GOP

In this part we suppose that we believe in a certain model for the GOP. It could be for example the Minimal Market Model described previously. Or

it could be an approximation of the GOP (by one of the indices described previously).
In any case, the GOP is assumed to be an intrinsic data. Then we can deduce a dynamics of zero-coupons. In the benchmark framework, the price $B^{GOP}(t,T)$ of a zero-coupon can be expressed in terms of the GOP under the historical probability:

$$B^{GOP}(t,T) = \mathbb{E}^{\mathbb{P}}\left[\frac{S_t^\theta}{S_T^\theta}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{G^*_{t,T}}|\mathcal{F}_t\right].$$

Again, if we believe in a model or an approximation for the GOP between dates t and T and call it $G^*_{t,T}$, then it is possible to deduce a yield curve for the GOP. The main difference with the Ramsey Rule is that here we do not need any further assumptions on utility functions. Using the definition of a yield curve, we get the following relation, under the historical probability:

$$R_T^{GOP}(s) = -\frac{1}{s}\ln \mathbb{E}^{\mathbb{P}}\left[\frac{1}{G^*_{T,T+s}}|\mathcal{F}_T\right]$$

This is a so called historical yield curve, implied from historical observable data on the GOP.
Again, we examine modifications of the yield curve in Chapters 3 to 5: in Chapter 3, in the case of an incomplete market and standard utility functions, and in Chapter 5 in the case of an incomplete market and dynamic utility functions.

Chapter 3

Growth Optimal Portfolio, long-term interest rates and utility maximization in an incomplete market

Contents

In this Chapter, our aim is to take into account incompleteness in a financial market, and to examine the consequences on the term structure of interest rates.

Several authors have considered expected utility maximization in an incomplete market with a fixed horizon $T > 0$. In [KS98] or [CK92], incompleteness comes from constraints on a portfolio. Pricing in an incomplete market has also been treated by [EQ95]. The incomplete market from a general point of view has been studied by He and Pearson [HP91], Karatzas, Lehoczky, Shreve and Xu [KLSX91], Kramkov and Schachermayer [KS99, KS03].

Let us first recall the main results presented in Chapter 2. We have treated the case of a complete financial market and established the Ramsey Rule in this case. An agent is acting in a complete market $\mathcal{M} = (r(.), b(.), \sigma(.), S(0))$, where $b(.)$ is a column vector of size N, $\sigma(.)$ an $N \times N$ matrix-valued process (these processes are assumed to satisfy the conditions given in Chapter 2), $S(0) = (S_0^1, \dots, S_0^N)^{\mathbf{T}}$ is a column vector of positive constants and $(W_t)_{0 \le t \le T}$ is a N-dimensional $\mathbb{P}$-brownian motion. We have obtained in Chapter 2 the following result on the optimal consumption process of the agent (2.6.2)

$$U_c^1(t, c_t^*) = U_c^1(0, c_0^*) \frac{1}{G_t^*},$$

Then, for a utility function of the form $U_1(t, c) = e^{-\beta t} u(c)$, by taking the expectation of the equation above, we get:

$$\exp(-\beta t) \mathbb{E}^{\mathbb{P}} \left[\frac{u'(c_t^*)}{u'(c_0^*)} \right] = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_0^t r_s ds \right) \right],$$

where $\mathbb{P}$ is the historical probability measure, and $\mathbb{Q}$ is the risk-neutral pricing probability. This leads to the Ramsey Rule, that is for all $0 \le T \le T^H$, the utility from consumption function $U_1(t, c) = e^{-\beta t} u(c)$ and the yield curve $R_0(T)$ are linked by the relation:

$$R_0(T) = \beta - \frac{1}{T} \log \mathbb{E}^{\mathbb{P}} \left[\frac{u'(c_T^*)}{u'(c_0^*)} \right].$$

We examine now what these results become in the incomplete market case. From now on, throughout this work, the incomplete market case is our framework of study (except Sections 4.2 and 4.3).

This Chapter is organized as follows. We describe first the incomplete market. Then we present utility maximization in an incomplete framework. We examine both the primal and the dual problem, and solutions to the primal and dual problem. Then we derive a new yield curve (corresponding to Davis prices) and we compare this to the yield curve obtained in Chapter 2. We also treat the case of zero-coupon bonds dynamics.

3.1 The incomplete market

In this Chapter, our framework is an incomplete financial market with a fixed horizon T. There are several ways to describe incompleteness in a financial market. The market we consider here is inspired by the one described in chapters 5 and 6 of [KS98]. Incompleteness comes from portfolio constraints. In particular, it is not possible to invest in some of the stocks. We consider an investment universe with N risky assets and assume that the investor can only invest in the riskless asset and in M risky assets among N, with $M < N$.

3.1.1 Framework: an incomplete market

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration of $\mathcal{F}$ satisfying the usual conditions that is, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$. We consider $W(.)$ an N-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, N being the total number of sources of uncertainty in the market.

We call $\tilde{\mathcal{M}}$ the market consisting of the riskless asset S^0 and of the first M tradable assets $S^1, \ldots, S^M$.

The price $S^0(.)$ of the riskless asset is given by $dS^0_t = S^0_t r_t dt$, where $r(.)$ is the spot rate, assumed to be a positive and $\mathbb{F}$-progressively measurable process.

There are M tradable risky assets (with $M < N$) in which the representative agent can invest. Their prices $S^i(.)$, for $i = 1, \ldots, M$ are assumed to be continuous Itô processes and have dynamics:

$$\frac{dS^i_t}{S^i_t} = \tilde{b}^i_t dt + \langle \tilde{\sigma}^i_t, dW_t \rangle,$$

where the drift process $\tilde{b}(.)$ is a $M \times 1$ column vector, and $\tilde{\sigma}(.)$ is a M by N volatility matrix, $\tilde{\sigma}^i(.)$ being its i-th line vector. The processes $\tilde{b}(.)$ and $\tilde{\sigma}(.)$ are assumed to be $\mathbb{F}$-progressively measurable processes. The matrix $\tilde{\sigma}\tilde{\sigma}^{\mathbf{T}}$ is assumed to be nonsingular (we recall that $^{\mathbf{T}}$ denotes the transpose of a matrix). We assume the existence of an $\mathbb{F}$-progressively measurable risk premium process $\tilde{\theta}(.)$, a column vector of dimension N such that for all $0 \leq t \leq T$ it satisfies:

$$\tilde{\theta}_t = \tilde{\sigma}_t^{\mathbf{T}}(\tilde{\sigma}_t\tilde{\sigma}_t^{\mathbf{T}})^{-1}(\tilde{b}_t - r_t\mathbf{1}_M),$$

where $\mathbf{1}_M$ is the M dimensional vector, where all components are equal to one. Thus $\tilde{\theta}_t$ is in the image of $\tilde{\sigma}_t^{\mathbf{T}}$. Integrability conditions on $\tilde{b}(.)$, $\tilde{\sigma}(.)$, $r(.)$ and $\tilde{\theta}(.)$ are assumed to be the same as in Chapter 2.

Consider a representative agent acting in the incomplete market $\tilde{\mathcal{M}}$. He starts from an initial wealth x. We call $\pi_t^i, i = 1, \ldots, M$ the fraction of his wealth invested at time t in each of the risky assets. For each date $0 \leq t \leq T$, the column vector of size M, $\pi_t = (\pi_t^1, \ldots, \pi_t^M)^{\mathbf{T}}$ is the portfolio process. The process π_t is assumed to be $\mathbb{F}$-progressively measurable, and satisfies the following condition:

$$\int_0^T ||\tilde{\sigma}_t^{\mathbf{T}}\pi_t||^2 dt < +\infty,$$

where T is the investment horizon. These fractions can be positive or negative but must always sum up to one, that is: $\sum_{j=0}^{M} \pi_t^j = 1$.

Remark 3.1.1 *Here we have considered a particular example of portfolio constraints, where the representative agent can not invest all the assets. More precisely, he can invest in M assets (with $M < N$) only. Then his portfolio is in $\mathfrak{K} = \mathbb{R}^M$.*
Another example would be an incomplete market with prohibition of short-selling. Then the fractions invested in the tradable asssets have to remain positive and $\mathfrak{K} = \mathbb{R}^{+M}$.
It is also possible to consider a more general form of constraints. For example the agent could invest in a convex cone $\mathfrak{K}$, see [KS98, KLSX91, HK04], or [Mra09].

The representative agent consumes a certain part of his wealth while he invests in the financial market, with a consumption rate c_t at time t. The consumption process is assumed to be $\mathbb{F}$-progressively measurable and such that $\int_0^T c_s ds < +\infty$ almost surely. For an agent starting from an initial wealth x and taking a portfolio process π and a consumption process c, his wealth at time t is given by $X_t^{x,c,\pi}$. We consider self-financing portfolios. We assume that we can substract a certain rate of consumption from these self-financing portfolios, at each time t. The wealth $X_t^{x,c,\pi}$ of the agent at time t is then a solution of:

$$dX_t^{x,c,\pi} = -c_t dt + X_t^{x,c,\pi} r_t dt + X_t^{x,c,\pi}\langle\tilde{\sigma}_t^{\mathbf{T}}\pi_t, dW_t + \tilde{\theta}_t dt\rangle, \text{ and } X_0^{x,c,\pi} = x.$$

Throughout this Chapter, we consider only wealth processes which are positive $\mathbb{P}$-almost surely for all $t \geq 0$. We say that (c, π) are admissible and denote $(c, \pi) \in \mathcal{A}(x)$ if (c, π) are $\mathbb{F}$ progressively measurable processes and:

$$X_t^{x,c,\pi} \geq 0, \forall t \in [0,T] a.s., \pi_t \in \mathfrak{K}, \forall t \in [0,T] a.s.$$

$\mathcal{A}(x)$ is the space of admissible portfolios.

We put in evidence here the important role of the volatility vector $\kappa_t := \tilde{\sigma}_t^{\mathbf{T}} \pi_t$. For the sake of simplicity, from now on we denote the wealth process by $X^{x,c,\kappa}(.)$, with dynamics:

$$dX_t^{x,c,\kappa} = -c_t dt + X_t^{x,c,\kappa} r_t dt + X_t^{x,c,\kappa} \langle \kappa_t, dW_t + \tilde{\theta}_t dt \rangle, \text{ and } X_0^{x,c,\kappa} = x.$$

We assume that the risk aversion of the representative agent is characterized by its preference structure (U^1, U^2), where U^1 and U^2 are continuous and twice differentiable functions on $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. The function $U^1(t, c)$ is the utility from consumption, and the function $U^2(x)$ is the utility from terminal wealth. We denote by $I_1(t, y)$ the inverse function of $U_c^1(t, c)$, the derivative of U^1 relatively to its second variable. We denote by $I_2(y)$ the inverse function of $U_x^2(x)$, the derivative of U^2 relatively to x.
For an agent with consumption rate $c(.)$, portfolio $\pi(.)$ (and volatility vector $\kappa_t := \tilde{\sigma}_t^{\mathbf{T}} \pi_t$) and preference structure (U^1, U^2), this problem is expressed as:

$$V(x) = \sup_{(c,\kappa) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^T U^1(s, c_s) ds + U^2(X_T^{x,c,\kappa})\right]. \tag{3.1.1}$$

In order to fix the ideas, we use the maximization problem from consumption and terminal wealth between dates 0 and T and denote by $V(x)$ the value function of this problem. Throughout this chapter, when we need to use the maximization problem from consumption only, or terminal wealth only, we denote respectively by V_1 and V_2 their value functions, as in Chapter 2.

3.1.2 Completion of the market

For the purpose of solving an expected utility maximization problem in an incomplete market where incompleteness comes from portfolio constraints, a dual process ν is introduced, it plays a role similar to Lagrange multipliers. The approach of [KS98] is to consider a family of auxiliary markets M_ν. For each dual process ν there is an auxiliary market $\mathcal{M}_\nu$. There are no longer portfolio constraints in this family of auxiliary markets. Each of these markets $\mathcal{M}_\nu$ is a way of completing the market $\tilde{\mathcal{M}}$.

We denote by $L = N - M$ the number of assets in which the agent cannot invest. One can fashion L fictitious assets in order to complete the market $\tilde{\mathcal{M}}$. In order to describe their dynamics, we define the matrix $\rho(.)$ of size $L \times N$, characterizing the volatility of these fictitious assets.
We choose the matrix ρ_t such that the columns of $\rho(.)$ are orthonormal vectors, and orthogonal to $\tilde{\sigma}_t$. This appears in the following equations: $\rho_t \rho_t^{\mathbf{T}} = I_L$, where I_L is the L by L identity matrix. And: $\tilde{\sigma}_t \rho_t^{\mathbf{T}} = 0$. We call σ_t the N by N matrix defined by:

$$\sigma_t = \begin{pmatrix} \tilde{\sigma}_t \\ \rho_t \end{pmatrix}.$$

And we can check that for any $t \in [0, T]$:

$$\sigma_t^{-1} = \begin{pmatrix} \tilde{\sigma}_t^{\mathbf{T}} (\tilde{\sigma}_t \tilde{\sigma}_t^{\mathbf{T}})^{-1} & \rho_t^{\mathbf{T}} \end{pmatrix}.$$

The dynamics of the assets in a market $\mathcal{M}_\nu$ are then:

$$dS_t^0 = S_t^0 r_t dt. \tag{3.1.2}$$

We call $S_t = (S_t^1, \ldots, S_t^M)^{\mathbf{T}}$ the vector of the M tradable risky assets prices, solution of the following SDE:

$$dS_t = S_t \Big(r_t \mathbf{1}_M dt + \tilde{\sigma}_t (\tilde{\theta}_t dt + dW_t) \Big), \ S_0 = (S_0^1, \ldots, S_0^M)^{\mathbf{T}}. \tag{3.1.3}$$

For all $0 \leq t \leq T$, we define the $\mathbb{F}$-progressively measurable process $\nu(.)$, called the dual process, which is a vector of size N. The vector S_t^ν of size L describes the prices of the fictitious assets:

$$dS_t^{(\nu)} = S_t^{(\nu)} \left(r_t \mathbf{1}_L dt + \rho_t (\nu_t dt + dW_t) \right). \tag{3.1.4}$$

We have the following relation:

$$Range(\tilde{\sigma}_t^{\mathbf{T}}) \oplus Ker(\tilde{\sigma}_t) = \mathbb{R}^N.$$

This leads us to define the orthogonal subsets $\mathcal{K}$ and $\mathcal{K}^\perp \subset \mathbb{R}^N$.
For all $0 \leq t \leq T$, by definition $\tilde{\theta}_t$ and κ_t are in the range of $\tilde{\sigma}_t^{\mathbf{T}}$, and we denote $\tilde{\theta}(.) \in \mathcal{K}$, $\kappa(.) \in \mathcal{K}$.
For all $0 \leq t \leq T$, ν_t is in $Ker(\tilde{\sigma}_t)$ (because $\tilde{\sigma}_t \nu_t = 0$) and we denote $\nu(.) \in \mathcal{K}^\perp$
We call $\theta^\nu(.)$ the risk premium process for the market $\mathcal{M}_\nu$:

$$\theta_t^\nu = \tilde{\theta}_t + \nu_t,$$

where ν_t is orthogonal to $\tilde{\theta}_t$.
We recall that we denote exponential martingales by:

$$\mathcal{E}(\int_0^t \langle \theta_s, dW_s \rangle) = \exp\left(\int_0^t \langle \theta_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\theta_s||^2 ds\right).$$

Following the notations from [KS98], we define Z_t^ν, a family of exponential local martingales.

$$Z_t^\nu = \mathcal{E}\big(-\int_0^t \langle \theta_s^\nu, dW_s \rangle\big) \tag{3.1.5}$$

$$= \mathcal{E}\big(-\int_0^t \langle \tilde{\theta}_s, dW_s \rangle\big)\mathcal{E}\big(-\int_0^t \langle \nu_s, dW_s \rangle\big) \tag{3.1.6}$$

In the previous expression, the first exponential does not depend on ν, that is, it does not depend on the completion of the market.

It is possible to define a family of probability measures $\mathbb{Q}^\nu$ on $\mathcal{F}_T$ by $\mathbb{Q}^\nu(A) = \mathbb{E}[Z_T^\nu \mathbf{1}_A]$, for all $A \in \mathcal{F}_T$. The process:

$$W_t^{\mathbb{Q}^\nu} = W_t + \int_0^t \theta_s^\nu ds,$$

for all $t \in [0, T]$ is a Brownian motion under $\mathbb{Q}^\nu$, relatively to the filtration $\mathbb{F}$.
We define the state price density process $H^\nu(.)$, for all $0 \le t \le T$ as:

$$H_t^\nu = \frac{Z_t^\nu}{S_t^0}. \tag{3.1.7}$$

The state price density process $H^\nu(.)$ is solution of the following stochastic differential equation, for all $\nu \in \mathcal{K}$, for all $0 \le t \le T$:

$$dH_t^\nu = -r_t dt - \langle \tilde{\theta}_t + \nu_t, dW_t \rangle,$$

where $\tilde{\theta}(.)$ and $\nu(.)$ are orthogonal, and $H_0^\nu = 1$.
We define the process $Y^{y,\nu}(.)$ such that for all $\nu \in \mathcal{K}$ and for all $y > 0$:

Definition 3.1.1 *For all* $0 \le t \le T$,

$$y \to Y_t^{y,\nu} = yH_t^\nu = y exp(-\int_0^t r_s ds - \int_0^t \langle \tilde{\theta}_s + \nu_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\tilde{\theta}_s + \nu_s||^2 du),$$

and $Y_0^{y,\nu} = y$. *This process appears naturally in the expression of the dual optimization problem. The function* $y \to Y_t^{y,\nu}$ *is linear relatively to its initial condition* y.

In the completed market $\mathcal{M}_\nu$ (including the assets in which the agent can invest and the fictitious assets) , the wealth process of an agent starting from initial wealth $x > 0$ is denoted $X^{\nu,x,c,\kappa}(.)$ and using equations (3.1.2,3.1.3, 3.1.4) its dynamics satisfies the following equation:

$$\frac{X_t^{\nu,x,c,\kappa}}{S_t^0} = x - \int_0^t \frac{c_s}{S_s^0} + \int_0^t \frac{X_s^{\nu,x,c,\kappa}}{S_s^0} \langle \kappa_s, dW_s^{\mathbb{Q}^\nu} \rangle. \tag{3.1.8}$$

We say that the $\mathbb{F}$-progressively measurable processes $(c, \kappa) \in \mathcal{A}^\nu(x)$ if $X_t^{x,c,\kappa,\nu} \geq 0$, for all t, $\mathbb{P}$-almost surely, and $\mathbb{E}[\int_t^T \min(0, U^1(s, c_s))ds] > -\infty$ and $\mathbb{E}[\min(0, X_T^{x,c,\kappa,\nu})] > -\infty$. Using (3.1.8), for all $\nu(.) \in \mathcal{K}^\perp$, and for all $0 \leq t \leq T$:

$$X_t^{\nu,x,c,\kappa} H_t^\nu + \int_0^t c_s H_s^\nu = x + \int_0^t X_s^{\nu,x,c,\kappa} H_s^\nu \langle \kappa_s - \theta_s^\nu, dW_s \rangle. \tag{3.1.9}$$

Thus for all $\nu(.) \in \mathcal{K}^\perp$, $X_t^{\nu,x,c,\kappa} H_t^\nu$ is a $\mathbb{P}$-local martingale, and this gives the budget constraint of the following section.

3.1.3 Optimization in incomplete markets

In [KS98], a way to solve this constrained problem is to introduce a fictitious completion of the market $\mathcal{M}_\nu$, parametrized by $\nu \in \mathcal{K}^\perp$. We solve first the auxiliary problem of maximizing the expected utility from both consumption and terminal wealth without regard to the porfolio constraint. That is, using the same notations as in Chapter 3, we solve the auxiliary maximization problem:

$$V^\nu(x) = \sup_{(c,\kappa) \in \mathcal{A}^\nu(x)} \mathbb{E}\left[\int_0^T U^1(t, c_t)dt + U^2(X_T^{\nu,x,c,\kappa})\right].$$

under the budget constraint:

$$\mathbb{E}\left[\int_0^T c_t H_t^\nu dt + H_T^\nu X_T^{\nu,x,c,\kappa}\right] \leq x,$$

The next step is to solve:

$$\inf_{\nu \in \mathcal{K}^\perp} V^\nu(x) \tag{3.1.10}$$

We assume that the optimum of this problem (3.1.10) is attained by a so-called optimal dual process $\nu^*(.)$. Then

$$V^{\nu^*}(x) = \inf_{\nu \in \mathcal{K}^\perp} V^\nu(x),$$

is the value function, solution of the maximization problem in an incomplete market (3.1.1).
From ν^*, we can deduce the corresponding process Z^{ν^*} and the probability $\mathbb{Q}^{\nu^*}$. The probablity measure $\mathbb{Q}^{\nu^*}$ is the optimal probability measure relative to a given utility function.

Proposition 3.1.1 *The optimal pricing probability $\mathbb{Q}^{\nu^*}$ can depend on the maturity, on the utility function (for exemple on β) and on y the wealth of the economy.*

This is studied in detail in Section 3.4.
It has been proved, for instance in [KS98] that if we assume the existence of an optimum ν^* for the utility maximization problem (3.1.10) in incomplete markets, the optimal consumption path is:

$$c_t^* = I_1(t, \mathcal{Y}_3(x) H_t^{\nu^*}),$$

where $\mathcal{Y}_3(x)$ is the Lagrange multiplier of the investment/consumption utility maximization problem defined in Chapter 2. The equation above can also be rewritten as:

$$U_c^1(t, c_t^*) e^{-\beta t} = U_c^1(0, c_0^*) H_t^{\nu^*}. \tag{3.1.11}$$

And the optimal terminal wealth is:

$$X_T^{*,x} = I_2(t, \mathcal{Y}_3(x) H_T^{\nu^*}),$$

And one can check that the process $X_t^{*,x} H_t^{\nu^*} + \int_0^t H_s^{\nu^*} c_s^* ds$ is a $\mathbb{P}$-martingale.

3.1.4 The dual point of view

This section is devoted to another approach of portfolio optimization: the approach by duality methods. This part is based on the work of Karatzas and Shreve [KS98] and Pham [Pha07]. We recall here the Fenchel transform of a concave function, then we define the dual optimization problem associated with the primal problem mentioned above.
We recall that Fenchel transforms of the utility from consumption function U^1 and utility from terminal wealth function U^2 are defined by:

$$\tilde{U}^1(t, y) = \inf_{c>0} \{U^1(t, c) - cy\}, \text{ for all } y > 0.$$

$$\tilde{U}^2(y) = \inf_{x>0} \{U^2(x) - xy\}, \text{ for all } y > 0.$$

The dual problem is, for a fixed $y > 0$, to minimize over $\nu \in \mathcal{K}^\perp$ the following expression:

$$\tilde{V}(y) = \inf_{\nu \in \mathcal{K}^\perp} \mathbb{E}[\int_0^T \tilde{U}^1(t, yH_t^\nu)dt + \tilde{U}^2(yH_T^\nu)].$$

When using the dual problem, the expression of the utility maximization no longer depends on consumption and wealth processes. Instead, the expression of the dual value function involves the process $Y_t^{y,\nu} = yH_t^\nu$, which is common to the term concerning the utility from consumption and the utility from terminal wealth. Thus, the parametrization by the process $Y_t^{y,\nu}$ appears naturally, for the dual optimization problem can be rewritten as:

$$\tilde{V}(y) = \inf_{\nu \in \mathcal{K}^\perp} \mathbb{E}\left[\int_0^T \tilde{U}^1(t, Y_t^{y,\nu})dt + \tilde{U}^2(Y_T^{y,\nu})\right],$$

The dual value function of the dual problem is then:

$$\tilde{V}(y) = \mathbb{E}\left[\int_0^T \tilde{U}^1(t, Y_t^{y,\nu^*})dt + \tilde{U}^2(Y_T^{y,\nu^*})\right]. \qquad (3.1.12)$$

The following proposition gives the relation between the value function and the dual value function:

Proposition 3.1.2 *From [Pha07]. For an utility function satisfying the assumptions given in Chapter* 2*, we have:*

$$\tilde{V}(y) = \sup_{x>0}\{V(x) - xy\}, \forall y > 0.$$

$$V(x) = \inf_{y>0}\{\tilde{V}(y) + xy\}, \forall x > 0.$$

3.1.5 Dynamic framework

The optimal processes $(c_t^*)_{t\geq 0}$ and $(X_t^{*,x})_{t\geq 0}$ are also solutions of the conditional utility maximization problem, between dates t and T such that $0 \leq t \leq T$:

$$esssup_{(c,\pi)\in\mathcal{A}(x)}\mathbb{E}\left[\int_t^T U^1(s, c_s)ds + U^2(T, X_T^{x,c,\kappa})|\mathcal{F}_t\right],$$

$$\text{s.c. } \mathbb{E}\left[\int_t^T H_{t,s}^\nu c_s ds + H_{t,T}^\nu X_T^{x,c,\kappa}|\mathcal{F}_t\right] \leq X_t^{*,x},$$

where for all $0 \leq s \leq t \leq T$, $H^{\nu}_{t,s} = H^{\nu}_s / H^{\nu}_t$. If the optimum ν^* is attained, the corresponding state price density process is $H^{\nu^*}_{t,s}$. Then the budget constraint at the optimum is, in this incomplete framework:

$$X_t^{*,x} = \mathbb{E}\left[\int_t^T H^{\nu^*}_{t,s} c^*_s ds + H^{\nu^*}_{t,T} X_T^{*,x} | \mathcal{F}_t\right].$$

And for the dual problem formulation:

$$essinf_{\nu \in \mathcal{K}^\perp} \mathbb{E}\left[\int_t^T \tilde{U}^1(s, Y^{y,\nu}_{t,s}) ds + \tilde{U}^2(Y^{y,\nu}_{t,T}) | \mathcal{F}_t\right],$$

where for all $\nu \in \mathcal{K}^\perp$, for all $0 \leq s \leq t \leq T$, the state price density process $Y^{y,\nu}_{t,s}$ is defined as: $Y^{y,\nu}_{t,s} = y H^{\nu}_s / H^{\nu}_t$.

In the following sections, we provide the expression of $\nu^*(.)$ for some simple examples of utility functions.

3.2 Logarithmic utility function: the GOP in incomplete markets

In the previous chapter we have presented the GOP (Growth Optimal Portfolio) and its properties in the case of a complete market $\mathcal{M}$. From now on, we consider the incomplete market described previously. We would like to express the dynamics of a Growth Optimal Portfolio in an incomplete market. More precisely, we put in evidence that the properties of the GOP in a complete market still hold here.

3.2.1 Solution of the utility maximization problem with a logarithmic utility function

Proposition 3.2.1 *For an incomplete market in the sense that we have defined, in the case of the logarithmic utility, the expression of the optimal state price density process $H^{\nu^*}_t$ is:*

$$H^{\nu^*}_t = exp\left(-\int_0^t r_s ds\right) exp\left(-\int_0^t \langle \tilde{\theta}_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\tilde{\theta}_s||^2 ds\right).$$

Proof. This is obtained by calculating the measure ν^* satisfying (3.1.10). For the logarithmic utility, the maximization of utility from terminal wealth problem is:

$$\sup_{\kappa} \mathbb{E}[\ln(X_T^{\nu,x,\kappa})] s.c. \mathbb{E}[H^{\nu}_T X_T^{\nu,x,\kappa}] \leq x.$$

The first order condition is: $\frac{1}{X_T^{\nu,x,\kappa}} = H_T^\nu \lambda$, where λ is the Lagrange multiplier. Using the constraint $\mathbb{E}[H_T^\nu X_T^{\nu,x,\kappa}] = x$ gives:

$$X_T^{\nu,x,\kappa} = \frac{x}{H_T^\nu}.$$

Then for any ν, the value function $V^\nu(x)$ is:

$$V^\nu(x) = \mathbb{E}\left[\ln\left(\frac{x}{H_T^\nu}\right)\right].$$

Hence, in order to satisfy (3.1.10), it remains to find:

$$\begin{aligned}
&\min_{\nu\in\mathcal{K}^\perp} \mathbb{E}\left[\ln(\frac{x}{H_T^\nu})\right] \\
&= \min_{\nu\in\mathcal{K}^\perp} \mathbb{E}\left[\int_0^t r_s ds + \int_0^t \langle\tilde{\theta}_s + \nu_s, dW_s\rangle + \int_0^t ||\tilde{\theta}_s + \nu_s||^2 ds\right] \\
&= \min_{\nu\in\mathcal{K}^\perp} \mathbb{E}\left[\int_0^t r_s ds + \int_0^t ||\tilde{\theta}_s + \nu_s||^2 ds\right]
\end{aligned}$$

Finally the expression to minimize for all $t \in [0,T]$ is:

$$\mathbb{E}\left[||\tilde{\theta}_t + \nu_t||^2\right].$$

The minimum is reached for $\nu^* \equiv 0$. With this choice of process $\nu^*(.)$, the expression of $H_t^{\nu^*}$ for the logarithmic utility is, by replacing in (3.3.3):

$$H_t^{\nu^*} = \exp\left(-\int_0^t r_s ds\right)\exp\left(-\int_0^t \langle\tilde{\theta}_s, dW_s\rangle - \frac{1}{2}\int_0^t ||\tilde{\theta}_s||^2 ds\right).$$

Hence the result.

■

In this case, the optimal wealth process for the logarithmic utility is:

$$X_T^{*,x} = x\exp\left(\int_0^t r_s ds\right)\exp\left(\int_0^t \langle\tilde{\theta}_s, dW_s\rangle + \frac{1}{2}\int_0^t ||\tilde{\theta}_s||^2 ds\right).$$

For utility functions other than the logarithmic utility, however, $H_t^{\nu^*}$ may depend on the chosen utility function, on the maturity and on y the wealth of the economy.

3.2.2 The GOP in incomplete markets: Maximizing the growth rate

In this section we use the first characterization of the GOP given by [PH06] in order to characterize the form of the GOP in an incomplete market, and check that it is consistent with the equation above.
Assuming that the dynamics of the assets are given by equations (3.1.2,3.1.3,3.1.4), we consider a self-financing portfolio without consumption, which value satisfies the SDE:

$$dS_t^\kappa = S_t^\kappa \big(r_t dt + \sum_{k=1}^{N} \sum_{j=1}^{M} \langle \kappa_t, \theta_t dt + dW_t \rangle \big).$$

As in Chapter 2, the growth rate is the drift in the SDE of the logarithm of $X_t^{x,\kappa}$:

$$g_t^\kappa = r_t + \langle \kappa_t, \tilde{\theta}_t \rangle - \frac{1}{2} ||\kappa_t||^2. \tag{3.2.1}$$

Definition 3.2.1 *We define the GOP in incomplete markets as the strictly positive portfolio maximizing the growth rate g_t^κ. Then the expression of the GOP is:*

$$\tilde{S}_t^\theta = \tilde{S}_0^\theta exp \left(\int_0^t r_s ds + \int_0^t \langle \tilde{\theta}_s, dW_s \rangle + \frac{1}{2} \int_0^t ||\tilde{\theta}_s||^2 dt \right), \tag{3.2.2}$$

with $\tilde{\theta}(.) \in \mathcal{K}$. One should notice that this process does not depend on the dual process $\nu(.) \in \mathcal{K}^\perp$.

Proof. First order conditions give the optimal fractions κ_t:

$$\kappa_t = \tilde{\theta}_t$$

And we obtain:

$$\tilde{S}_t^\theta = \tilde{S}_0^\theta \exp \left(\int_0^t r_s ds + \int_0^t \langle \tilde{\theta}_s, dW_s \rangle + \frac{1}{2} \int_0^t ||\tilde{\theta}_s||^2 dt \right).$$

We call this process $\tilde{S}_t^\theta$ in order to avoid confusion with the GOP in a complete market.
■

Thus the GOP in the incomplete market $\tilde{\mathcal{M}}$ is a tradable asset: it depends only on the first M risky assets and on the riskless asset. It is also the inverse of the optimal state price density for the logarithmic utility $H_t^{\nu^*}$.

3.3 Towards the Ramsey Rule in incomplete markets

In this section, we denote $\nu^*(y)$ for the optimal dual process in order to underline the fact that the optimal dual process depends on y, the wealth in the economy.

3.3.1 Relation between marginal utility from consumption and GOP in an incomplete market

The formula linking the optimal consumption path $c^{\nu^*(y)}$ and the state price density process $H^{\nu^*(y)}(.)$ is

$$U^1_c(t, c_t^{\nu^*(y)})e^{-\beta t} = U^1_c(0, c_0^{\nu^*(y)})H_t^{\nu^*(y)}. \tag{3.3.1}$$

In particular if we take $U^1(t,x) = e^{-\beta t}u(x)$, 3.1.11 becomes:

$$u'(c_t^{\nu^*(y)})e^{-\beta t} = u'(c_0^{\nu^*(y)})H_t^{\nu^*(y)}. \tag{3.3.2}$$

On the other hand, equation (3.1.11) and the definition of $H_t^{\nu^*(y)}$ show that:

$$e^{-\beta t}\frac{u'(c_t^{\nu^*(y)})}{u'(c_0^{\nu^*(y)})} \tag{3.3.3}$$

$$= H_t^{\nu^*(y)} \tag{3.3.4}$$

$$= \exp\left(-\int_0^t r_s ds\right)\mathcal{E}\left(-\int_0^t \langle\tilde{\theta}_s, dW_s\rangle\right)\mathcal{E}\left(-\int_0^t \langle\nu_s, dW_s\rangle\right). \tag{3.3.5}$$

In this last expression, we recall that ν and $\tilde{\theta}$ are orthogonal.

This result is analoguous to equation (2.6.2), except that H_t^0 (the inverse of the GOP) has been replaced by $H_t^{\nu^*(y)}$. The difference however is that $H_t^{\nu^*(y)}$ depends on the chosen utility function u.
Using equation (3.1.11) and taking the expectation of each side of this equation under the probability $\mathbb{P}$, then using the definition of the probability $\mathbb{Q}^{\nu^*(y)}$ we obtain:

$$e^{-\beta t}\mathbb{E}\left[\frac{u'(c_t^{\nu^*(y)})}{u'(c_0^{\nu^*(y)})}\right] = \mathbb{E}\left[H_t^{\nu^*(y)}\right] = \mathbb{E}\left[e^{-\int_0^t r_s ds}Z_t^{\nu^*(y)}\right] \tag{3.3.6}$$

$$= \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}\left[e^{-\int_0^t r_s ds}\right]. \tag{3.3.7}$$

In the complete market case this expression was equal to the zero-coupon price, that is the expectation of $\exp\left(-\int_0^t r_s ds\right)$ under the risk-neutral probability, which was unique. But in the incomplete market case instead, we need to find the expectation of $\exp\left(-\int_0^t r_s ds\right)$ under the probability $\mathbb{Q}^{\nu^*(y)}$, which may depend on the chosen utility function and on maturity T.

3.3.2 Davis prices

In a complete market, zero-coupon prices (under the risk-neutral probability) are well-defined. This is no longer the case here.
In this section, we interpret the expression obtained above:

$$\mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}\left[e^{-\int_0^t r_s ds}\right], \tag{3.3.8}$$

in terms of Davis prices.
A way to price the expression (3.3.8) is to use the method studied by Davis [Dav98], to value options for an agent endowed with a particular utility function. Davis defines the price of a contingent claim (with payoff Φ_T at time T) in an incomplete market using a marginal rate. Let us assume that this contingent claim is traded at price p at date 0. An inverstor with initial wealth x invests an amount δ in this contingent claim. Then his final wealth is: $X^{x-\delta,c,\kappa} + \frac{\delta}{p}\Phi_T$. Assuming that his preference structure is (U^1, U^2), he seeks to maximize the following investment program:

$$W(\delta, x, p) = \sup_{(c,\kappa)} \mathbb{E}[U^2(X^{x-\delta,c,\kappa} + \frac{\delta}{p}\Phi_T) + \int_0^T U^1(t, c_t)dt]$$

Definition 3.3.1 *Assume that the equation:*

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0,$$

has a unique solution p_0^*. Then p_0^* is the Davis price of the contingent claim at time $t = 0$.

This Davis price holds for relatively small amounts. This is also the smallest price possible. Bigger nominal induce more risk and lead to higher prices.
In the original definition given in [Dav98] there is no consumption process, but one can check that taking into account the consumption process (as we do here) does not change the definition of p_0^*, because there is no δ in the consumption term.

We remind that $X_T^{*,x}$ is the optimal wealth at time T of an agent starting from initial wealth x at time 0 and that $V'(x)$ is the derivative of the value function $V(x)$ of the consumption/investment problem relatively to x. The following theorem comes from [Dav98]:

Theorem 3.3.1 *For a given payoff Φ_T at time T, the Davis price at time 0 is given by:*

$$p_0^* = \frac{\mathbb{E}[U_x^2(X_T^{*,x})\Phi_T]}{V'(x)}, \tag{3.3.9}$$

Our purpose is to rewrite this expression in order to put into evidence the term: $\mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_0^T r_s ds)]$.

Proposition 3.3.1 *In an incomplete market, for a given payoff Φ_T at time T, the Davis price at time 0 can be rewritten as:*

$$p_0^* = \mathbb{E}[H_T^{\nu^*(y)}\Phi_T]. \tag{3.3.10}$$

Proof. In the incomplete market case, the terminal wealth corresponding to the optimal consumption/investment strategy of an agent is: $X_T^* = I_2(\mathcal{Y}_3(x)H_T^{\nu^*(y)})$, where the Lagrange multiplier $\mathcal{Y}_3(x)$ is the inverse of $\mathcal{X}_3$ (2.5.5). Similarly the optimal consumption path at time t is given by: $c_t^* = I_1(\mathcal{Y}_3(x)H_t^{\nu^*(y)})$. Hence the value function $V(x)$ is:

$$V(x) = \mathbb{E}[U^2(I_2(\mathcal{Y}_2(x)H_T^{\nu^*(y)})) + \int_0^T U^1(I_1(\mathcal{Y}_3(x)H_t^{\nu^*(y)}))dt].$$

Deriving this expression relatively to x and using the fact that U_x^2 is the inverse of I_2, U_x^1 is the inverse of I_1 gives:

$$\begin{aligned}
V'(x) &= \mathcal{Y}_3(x)\mathbb{E}[H_T^{\nu^*(y)}\frac{d}{dx}I_2(\mathcal{Y}_3(x)H_T^{\nu^*(y)})] + \mathcal{Y}_3(x)\mathbb{E}[\int_0^T H_t^{\nu^*(y)}\frac{d}{dx}I_1(\mathcal{Y}_3(x)H_t^{\nu^*(y)})dt] \\
&= \mathcal{Y}_3(x)\mathbb{E}[H_T^{\nu^*(y)}I_2'(\mathcal{Y}_3(x)H_T^{\nu^*(y)})]\mathcal{Y}_3'(x) + \mathcal{Y}_3(x)\mathbb{E}[\int_0^T H_t^{\nu^*(y)}I_1'(\mathcal{Y}_3(x)H_t^{\nu^*(y)})dt]\mathcal{Y}_3'(x) \\
&= \mathcal{Y}_3(x)(\mathcal{X}_2 \circ \mathcal{Y}_3)'(x) + \mathcal{Y}_3(x)(\mathcal{X}_1 \circ \mathcal{Y}_3)'(x) \\
&= \mathcal{Y}_3(x),
\end{aligned}$$

where we use the fact that $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}_3$. This is a classical result which can be found for instance in [KS98].
Furthermore, the other term in Theorem above can be rewritten as:

$$\mathbb{E}[U_x^2(X_T^*)\Phi_T] = \mathcal{Y}_3(x)\mathbb{E}[H_T^{\nu^*(y)}\Phi_T].$$

Thus, in an incomplete market, the Davis price p_0^* at time 0 is given by:

$$p_0^* = \mathbb{E}[H_T^{\nu^*(y)} \Phi_T]. \tag{3.3.11}$$

where the expectations are taken under the historical probability.
■

In particular if the pay-off at time T is $\Phi_T = 1$, the Davis price is given by:

$$p_0^* = \mathbb{E}[H_T^{\nu^*(y)}] = \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_0^T r_t dt)].$$

Thus p_0^* is the price to pay at time 0 in order to have a certain pay-off of 1 at time T. The expression we wanted to interpret in this section corresponds to a zero-coupon price in an incomplete market.
Thus we assume that the market prices zero-coupons bonds with Davis prices. This is only true for small amounts of wealth, and very dependent on the chosen utility function.
In the following, we denote, for the zero-coupon price at time 0 with maturity T deduced from Davis prices:

$$B^{\nu^*(y)}(0,T) = p_0^* = \mathbb{E}[H_T^{\nu^*(y)}] = \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_0^T r_t dt)]. \tag{3.3.12}$$

More generally, for all $0 \leq t \leq T$, we denote by $B^{\nu^*(y)}(t,T)$ zero-coupon bond prices at time t with maturity T:

$$B^{\nu^*(y)}(t,T) = \mathbb{E}[H_{t,T}^{\nu^*(y)}] = \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t]. \tag{3.3.13}$$

We also call $R_0^{\nu^*(y)}(T)$ the rate of return at time T of a certain amount of money invested at time 0, that is the rate such that an amount of $e^{-TR_0^{\nu^*(y)}(T)}$ invested at time 0 gives a certain pay-off of 1 at time T. Then:

$$R_0^{\nu^*(y)}(T) = -\frac{1}{T}\log B^{\nu^*(y)}(0,T) = -\frac{1}{T}\log \mathbb{E}^{\mathbb{P}}[H_T^{\nu^*(y)}] = -\frac{1}{T}\log \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_0^T r_s ds)]. \tag{3.3.14}$$

The Davis price holds for relatively small amounts. Hence we have defined a yield curve in incomplete markets $R_0^{\nu^*(y)}(T)$, corresponding to Davis prices, which is accurate for small amounts of money only.

Remark 3.3.1 *More generally, between dates $t \leq T$, the yield curve is given by:*

$$R_T^{\nu^*(y)}(s) = -\frac{1}{s}\log \mathbb{E}^{\mathbb{P}}[H_{T,T+s}^{\nu^*(y)}] = -\frac{1}{s}\log \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[exp(-\int_T^{T+s} r_s ds)|\mathcal{F}_T]. \tag{3.3.15}$$

3.3.3 The Ramsey Rule in incomplete markets

Proposition 3.3.2 *For a choice of utility function $U^1(t,c) = e^{-\beta t}u(c)$, using Davis prices, we can deduce from the previous paragraphs an analogue of the Ramsey Rule rule in incomplete markets, linking $R_0^*(T)$ and the optimal consumption trajectory $c^{\nu^*(y)}(.)$, for all $0 \le t \le T$:*

$$R_0^{\nu^*(y)}(T) = \beta - \frac{1}{T}\log \mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^{\nu^*(y)})}{u'(c_0^{\nu^*(y)})}\right]. \tag{3.3.16}$$

Proof. In (3.3.6), we have:

$$e^{-\beta t}\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_t^{\nu^*(y)})}{u'(c_0^{\nu^*(y)})}\right] = \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}\left[e^{-\int_0^t r_s ds}\right].$$

The use of equation (3.3.15) gives the result.

■

Let us start with a remark concerning this yield curve corresponding to Davis prices. As we have underlined with the notation $\nu^*(y)$, the optimal dual process depends on y, the wealth in the economy. Thus the yield curve given in the equation above depends on the wealth, which is a reasonable assumption for an economic model.

For a time separable utility from consumption function of the form $U^1(t,y) = \exp(-\beta t)u(y)$, the Ramsey rule in a complete market was:

$$R_0(T) = \beta - \frac{1}{T}\mathbb{E}\left[\frac{u'(c_T^*)}{u'(c_0^*)}\right],$$

and in an incomplete market:

$$R_0^{\nu^*(y)}(T) = \beta - \frac{1}{T}\mathbb{E}\left[\frac{u'(c_T^{\nu^*(y)})}{u'(c_0^{\nu^*(y)})}\right],$$

have relatively similar forms and differ only through the expression of the optimal consumption. For this choice of time separable utility function, the parameter β has a strong impact on the form of the yield curve.

Remark 3.3.2 *From the point of view of the economists, the optimal consumption $c_t^{\nu^*(y)}$ is known. The utility function is known. The historical probability is known. From this it is possible to deduce the dynamics of the interest rates and the properties of $\mathbb{Q}^{\nu^*(y)}$.*

3.3.4 Limits of the Ramsey Rule

We have started our study of long term interest rates with a focus on the Ramsey Rule: we have established the Ramsey Rule in a complete then in an incomplete market. But it is obtained with a Davis price, which is a marginal price, obtained for small amounts of money. Thus the Ramsey Rule gives a certain approximation of the rate, a certain benchmark. But it is not the rate that would be used for transactions for bigger amounts.
For an expression which would depend less on the amount invested, a pricing by indifference could be used.

3.4 Power utility functions: a detailed example

3.4.1 Expression of V_ν with a power utility function

Here we provide more details about the case of the power utility. Let assume that the representative agent has a preference structure (U^1, U^2) such that:

$$U^1(t,x) = U^2(x) = \frac{x^\alpha}{\alpha},$$

for $0 < \alpha < 1$. The maximization of utility from terminal wealth problem is:

$$\sup_\kappa \mathbb{E}\left[\frac{(X_T^{\nu,x,\kappa})^\alpha}{\alpha}\right] s.c. \mathbb{E}[H_T^\nu X_T^{\nu,x,\kappa}] \leq x,$$

where $\alpha < 1$. From the first order condition we deduce that:

$$X_T^{\nu,x,\kappa} = (H_T^\nu \lambda)^{\frac{1}{\alpha-1}},$$

where λ is the Lagrange multiplier. Using the constraint $\mathbb{E}[H_T^\nu X_T^{\nu,x,\kappa}] = x$ gives:

$$X_T^{\nu,x,\kappa} = \frac{x}{\mathbb{E}\left[(H_T^\nu)^{\frac{1}{\alpha-1}}\right]}(H_T^\nu)^{\frac{1}{\alpha-1}}.$$

Then for any $\nu \in \mathcal{K}^\perp$, the value function $V_2^\nu(x)$ is:

$$\begin{aligned} V_2^\nu(x) &= \mathbb{E}\left[\frac{x^\alpha}{\alpha}\frac{(H_T^\nu)^{\frac{\alpha}{\alpha-1}}}{\left(\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}}\right]\right)^\alpha}\right] \\ &= \frac{x^\alpha}{\alpha}\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}}\right]^{1-\alpha} \end{aligned}$$

The minimization problem on ν is then:

$$\inf_{\nu\in\mathcal{K}^\perp} V_2^\nu(x) = \inf_{\nu\in\mathcal{K}^\perp} \frac{x^\alpha}{\alpha}\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}}\right]^{1-\alpha}. \tag{3.4.1}$$

For the problem of maximizing the utility from consumption only (Problem 1) with a utility function $U^1(t,x) = x^\alpha/\alpha$ in an incomplete market between 0 and T, one can check with similar calculations that the problem becomes:

$$\inf_{\nu\in\mathcal{K}^\perp} V_1^\nu(x) = \inf_{\nu\in\mathcal{K}^\perp} \frac{x^\alpha}{\alpha}\mathbb{E}\left[\int_0^T (H_t^\nu)^{\frac{\alpha}{\alpha-1}} dt\right]^{1-\alpha}. \tag{3.4.2}$$

And for the problem of maximizing the utility from consumption and terminal wealth between dates 0 and T, with a preference structure $U^1(t,x) = U^2(x) = x^\alpha/\alpha$ the problem becomes:

$$\inf_{\nu\in\mathcal{K}^\perp} V^\nu(x) = \inf_{\nu\in\mathcal{K}^\perp} \frac{x^\alpha}{\alpha}\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}} + \int_0^T (H_t^\nu)^{\frac{\alpha}{\alpha-1}} dt\right]^{1-\alpha}. \tag{3.4.3}$$

For this choice of utility function, in an incomplete market such as we have defined it throughout Chapter 3, the optimal dual process ν depends on the chosen utility function and on maturity T. This will be detailed in the following subsection, where we provide an explicit example, in order to show an example of incomplete market where the yield curve is modified.

3.4.2 Example of interest rate model

Here we consider an example of incomplete market with 2 sources of uncertainty ($N = 2$). Let $W_t = (W_t^1, W_t^2)^{\mathbf{T}}$ be a N-dimentional $\mathbb{P}$-Brownian motion, thus W_t^1 and W_t^2 are independant standard Brownian motion. The representative agent can invest in the riskless asset with dynamics $dS_t^0 = r_t S_t^0 dt$ and in one risky asset ($M = 1$) with price S_t at time t:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_1(dW_t^1 + \theta_t^1 dt),$$

where $\sigma_1 > 0$ is a constant and θ_t^1 is the risk premium process. The state price density process $H^\nu(.)$ has dynamics:

$$\frac{dH_t^\nu}{H_t^\nu} = -r_t dt + (\nu_t dW_t^2 + \theta_t^1 dW_t^1)$$

Thus for all $0 \leq \alpha \leq 1$, we have:

$$(H^\nu)_T^{\frac{\alpha}{\alpha-1}} = \exp\left(\frac{\alpha}{\alpha-1}\left(-\int_0^T r_s ds + \int_0^T \nu_s dW_s^2 + \int_0^T \theta_s^1 dW_s^1 - \frac{1}{2}\int_0^T (||\nu_s||^2 + ||\theta_s^1||^2) ds\right)\right).$$

In addition to that, we assume that short rate dynamics follows an Ornstein-Uhlenbeck process:

$$dr_t = a(b - r_t)dt - \sigma dW_t^2.$$

Thus here the dynamics of r_t depends only on $W^2(.)$:

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) - \sigma \int_0^t e^{a(t-s)} dW_s^2.$$

And thus we deduce:

$$\int_0^T r_s ds = bT + \frac{r_0 - b}{a}(1 - e^{-aT}) - \frac{\sigma}{a}\int_0^T (1 - e^{a(T-s)})dW_s^2.$$

One can check that this last expresion can be rewritten as:

$$\int_0^T r_s ds = \int_0^T \mathbb{E}[r_s]ds - \int_0^T \Sigma_{s,T} dW_s^2,$$

where for all $0 \leq s \leq T$:

$$\Sigma_{s,T} = \frac{\sigma}{a}(1 - e^{a(T-s)}).$$

We refer to the Introduction for a presentation of the classic Vasicek model in a complete market.
We recall that exponential martingales are denoted by:

$$\mathcal{E}(\int_0^t \theta_s^1 dW_s^1) = \exp(\int_0^t \theta_s^1 dW_s^1 - \frac{1}{2}\int_0^t ||\theta_s^1||^2 ds)$$

Thus one obtains:

$$\begin{aligned}
&\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}}\right] \\
&= \mathbb{E}\left[\mathcal{E}(\frac{\alpha}{\alpha-1}\int_0^T \theta_s^1 dW_s^1)\mathcal{E}(\frac{\alpha}{\alpha-1}\int_0^T (\nu_s + \Sigma_{s,T})dW_s^2)\right. \\
&\left.\times \exp\left(-\frac{\alpha}{\alpha-1}\int_0^T (\mathbb{E}[r_s] + \frac{1}{2}(||\theta_s^1||^2 + ||\nu_s||^2))ds + \frac{\alpha^2}{2(\alpha-1)^2}\int_0^T \theta_s^1 + ||\nu_s + \Sigma_{s,T}||^2 ds\right)\right].
\end{aligned}$$

As we minimize over ν in:

$$\mathbb{E}\left[(H_T^\nu)^{\frac{\alpha}{\alpha-1}}\right],$$

we thus calculate:

$$\inf_{\nu \in \mathcal{K}^\perp}(-\frac{\alpha}{2(\alpha-1)}||\nu_s||^2 + \frac{\alpha^2}{2(\alpha-1)^2}||\nu_s + \Sigma_{s,T}||^2)$$

Thus we find for all $0 \leq s \leq T$:

$$\nu_s^* = -\alpha \Sigma_{s,T} = -\alpha \frac{\sigma}{a}(1 - e^{a(T-s)}).$$

Hence we see that the state price density process H^{ν^*} and the new yield curve depend on the chosen utility function (through the parameter α) and on maturity T (but not on y, the wealth of the economy).

3.5 Dynamics of zero coupon bonds

Let us recall that zero coupon bond prices deduced from Davis prices are denoted $B^{\nu^*(y)}(t,T)$ and given by:

$$B^{\nu^*(y)}(t,T) = \mathbb{E}^{\mathbb{P}}[H_T^{\nu^*(y)} | \mathcal{F}_t]$$

In this framework, the dynamics of zero-coupons $B^{\nu^*(y)}(t,T)$ given by Davis prices is a solution of the following SDE:

$$\frac{dB^{\nu^*(y)}(t,T)}{B^{\nu^*(y)}(t,T)} = r_t dt + \langle \Gamma^{\nu^*(y)}(t,T), dW_t + \theta_t^{\nu^*(y)} dt \rangle, \; B^{\nu^*(y)}(T,T) = 1. \quad (3.5.1)$$

This can be proven along the same lines as in Chapter 2. In the particular case where $\nu^*(y) \equiv 0$, the associated zero-coupon bond depends only on the tradeable assets, it is given by the expectation of the GOP in an incomplete market(see expression below), this is why we denote it B^{GOP}.

$$B^{GOP}(t,T) = \mathbb{E}[e^{-\int_t^T r_s ds} \mathcal{E}(-\int_0^t \langle \tilde{\theta}_s, dW_s \rangle) | \mathcal{F}_t].$$

For all $0 \leq t \leq T$ the zero coupon bond price B^{GOP} satisfies:

$$\exp(-\int_0^T r_s ds) \mathcal{E}\big(-\int_0^T \langle \tilde{\theta}_s, dW_s \rangle \big) = B^{GOP}(0,T) \mathcal{E}\big(\int_0^T \langle \Gamma^{GOP}(s,T), dW_s \rangle \big). \quad (3.5.2)$$

This expression is useful in the following proposition.

Proposition 3.5.1 *It is possible to express $B^{\nu^*(y)}(t,T)$ as a function of $B^{GOP}(t,T)$, for all $0 \leq t \leq T$.*

$$B^{\nu^*(y)}(t,T) = B^{GOP}(t,T) \mathbb{E}^{\mathbb{Q}^{GOP}} \left[\mathcal{E} \left(-\int_t^T \langle \nu^*(y)_s, dW_s \rangle \right) | \mathcal{F}_t \right],$$

where $\mathbb{Q}_T^{GOP}$ is the probability measure which density relatively to $\mathbb{P}$ is given by:

$$\frac{d\mathbb{Q}_T^{GOP}}{d\mathbb{P}} = \frac{e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T \langle\tilde{\theta}_s, dW_s\rangle\right)}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T \langle\tilde{\theta}_s, dW_s\rangle\right)\right]}.$$

Proof. Using the expression of $B^{GOP}(t,T)$, we find:

$$\begin{aligned} B^{\nu^*(y)}(t,T) &= B^{GOP}(t,T)\frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}\mathcal{E}\left(-\int_t^T \langle\tilde{\theta}_s, dW_s\rangle\right)\mathcal{E}\left(-\int_t^T \langle\nu^*(y)_s, dW_s\rangle\right)|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}\mathcal{E}\left(-\int_t^T \langle\tilde{\theta}_s, dW_s\rangle\right)|\mathcal{F}_t\right]} \\ &= B^{GOP}(t,T)\mathbb{E}^{\mathbb{Q}_T^{GOP}}\left[\mathcal{E}\left(-\int_t^T \langle\nu^*(y)_s, dW_s\rangle\right)|\mathcal{F}_t\right], \end{aligned}$$

where we call $\mathbb{Q}_T^{GOP}$ the probability which density relatively to $\mathbb{P}$ is:

$$\frac{d\mathbb{Q}_T^{GOP}}{d\mathbb{P}} = \frac{e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T \langle\tilde{\theta}_s, dW_s\rangle\right)}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r_s ds}\mathcal{E}\left(-\int_0^T \langle\tilde{\theta}_s, dW_s\rangle\right)\right]}.$$

The definition of this probability measure is inspired by the definition of a forward neutral probability measure (see the end of Chapter 4).

■

With a power utility function, along the same lines as in the previous section, and using also the information on zero-coupon bond dynamics, it is possible to deduce more precisely how the yield curve is modified.

Proposition 3.5.2 *With the choice of optimal dual process $\nu^*(y)(.)$ presented above*

$$R^{\nu^*(y)}(0,T) = R^{GOP}(0,T) - \frac{\alpha}{T}\int_0^T ||\Gamma^{GOP}(s,T)||^2 ds.$$

Proof. Using equation (3.5.2), and for all $\nu(.) \in \mathcal{K}^{\perp}$:

$$H_T^{\nu} = B^{GOP}(0,T)\mathcal{E}\big(\int_0^T \langle\Gamma^{GOP}(s,T), dW_s\rangle\big)\mathcal{E}\big(-\int_0^T \langle\nu_s, dW_s\rangle\big). \qquad (3.5.3)$$

Then in the case of a power utility function $U^2(x) = \frac{x^\alpha}{\alpha}$, with $0 < \alpha < 1$, the optimal dual process $\nu^*(y)$ is obtained by minimizing $\mathbb{E}[(H_T^{\nu})^{\frac{\alpha}{\alpha-1}}]$ over all $\nu(.) \in \mathcal{K}^{\perp}$. This gives an optimal dual process $\nu_t^* = -\alpha\Gamma^{GOP}(t,T)$. Thus:

$$\mathbb{E}[H_T^{\nu^*(y)}] = B^{GOP}(0,T)\exp\big(\int_0^T \alpha||\Gamma^{GOP}(s,T)||^2 ds\big).$$

Dividing this equation by T and taking the logarithm gives the result.
■

We explore again zero-coupon dynamics in the case of dynamic utility functions in Chapter 5, and put into evidence how the use of dynamic utility functions allows more freedom degrees, and how the dependance with maturity diseappears in the case of dynamic utility functions.

Let us now conclude this Chapter with a remark concerning forward rates in this context.

Remark 3.5.1 *The EDS satisfied by zero coupon dynamics is:*

$$\begin{aligned} dB^{\nu^*(y)}(t,T) &= B^{\nu^*(y)}(t,T)(r_t dt + \langle \Gamma^{\nu^*(y)}(t,T), \theta_t^{\nu^*(y)} + dW_t \rangle) \\ &= B^{\nu^*(y)}(t,T)(r_t dt + \langle \Gamma^{\nu^*(y)}(t,T), dW_t^{\nu^*(y)} \rangle), \end{aligned}$$

where we recall that the Brownian motion $(W_t^{\nu^*(y)})_{0\leq t\leq T}$ *is defined by:*

$$W_t^{\nu^*(y)} = \int_0^t \theta_s^{\nu^*(y)} + W_t.$$

That is:

$$\begin{aligned} B^{\nu^*(y)}(t,T) = \frac{B^{\nu^*(y)}(0,T)}{B^{\nu^*(y)}(0,t)} exp\big(&\int_0^t \langle \Gamma^{\nu^*(y)}(s,T) - \Gamma^{\nu^*(y)}(s,t), dW_s^{\nu^*(y)} \rangle \\ &- \frac{1}{2}\int_0^t (||\Gamma^{\nu^*(y)}(s,T)||^2 - ||\Gamma^{\nu^*(y)}(s,t)||^2) ds \big) \end{aligned}$$

Forward rates are $f^{\nu^*(y)}(t,T)$ *defined by:*

$$f^{\nu^*(y)}(t,T) = -\frac{\partial}{\partial T} \ln B^{\nu^*(y)}(t,T).$$

$$f^{\nu^*(y)}(t,T) = f^{\nu^*(y)}(0,T) - \int_0^T \langle \frac{\partial}{\partial T}\Gamma^{\nu^*(y)}(s,T), -\Gamma^{\nu^*(y)}(s,T) + dW_s^{\nu^*(y)} \rangle.$$

Thus we obtain an HJM-like equation

$$r_t = f^{\nu^*(y)}(t,t) = f^{\nu^*(y)}(0,t) - \int_0^t \langle (\frac{\partial}{\partial T}\Gamma^{\nu^*(y)}(s,T))_{T=t}, -\Gamma^{\nu^*(y)}(s,t) + dW_s^{\nu^*(y)} \rangle.$$

The same remarks concerning the fact that these expressions depend on Davis prices, (which hold for small amounts only), apply here.

Chapter 4

Two extensions: random horizon and random initial consumption

Contents

The purpose of this Chapter is to give two extensions of the previous Chapters. First, we show that the expected utility maximization problem from consumption and terminal wealth in the $\mathbb{F}$-market can be interpreted as an expected utility maximization problem from terminal wealth only, in a new $\mathbb{G}$-market which we define.
Then we treat the case where there is uncertainty on a parameter of the model, a reasonable assumption in the economic reality. Wee see how this case corresponds to a random initial consumption.
These two extensions of the classical expected utility maximization problem correspond each to a filtration enlargement, this is why we put them together in this Chapter. In the first case it is a progressive filtration enlargement. It comes from the introduction of a random horizon ζ. In the second case it is an initial filtration enlargement, corresponding to a random initial consumption.

4.1 The $\mathbb{G}$-market: a new point of view on consumption

The first part of this Chapter is organized as follows. First we show that we can unify consumption and wealth. More precisely, we show that it is possible to replace the maximization of expected utility problem from consumption and terminal wealth by a maximization of expected utility from terminal wealth with a random horizon. The consumption process can be seen as a certain quantity of wealth. Thus instead of solving a maximization of expected utility problem from consumption and terminal wealth we can solve a maximization of expected utility from terminal wealth only.
A similar approach is chosen for the dual expected utility maximization problem. It is more naturally parametrized by the Lagrange multiplier, a simple function of the initial consumption. Thus, if we want to put the emphasis on consumption and initial consumption, the problem that is the best suited to our study is the dual problem. This is the reason why we priviledge the dual problem.
In this Chapter we use several references concerning the theory of stochastic processes and filtration enlargement. In particular we refer to [Jac79, DM75, Jea]. We also use several references concerning utility maximization in incomplete market, such as [KS98, Pha07, HK04].

4.1.1 Framework: an incomplete market

In this part of the Chapter, our framework is an incomplete financial market with a fixed horizon T, such as the one described in Chapter 3. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration of $\mathcal{F}$ satisfying the usual conditions that is, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$. We consider $W(.)$ an N-dimensional standard Brownian motion defined on $(\Omega, \mathbb{F}, \mathbb{P})$, N being the total number of sources of uncertainty in the market. In this financial market there is one riskless asset with price $S^0(.)$ given by $dS^0_t = S^0_t r_t dt$, where $r(.)$ is the spot rate, assumed to be a positive and $\mathbb{F}$-progressively measurable process. There are M tradable risky assets (with $M < N$) in which the representative agent can invest. Their prices $S^i(.)$, for $i = 1, \ldots, M$ are assumed to be continuous Itô processes and have dynamics:

$$\frac{dS^i_t}{S^i_t} = \tilde{b}^i_t dt + \langle \tilde{\sigma}^i_t, dW_t \rangle,$$

where the drift process $\tilde{b}(.)$ is a $M \times 1$ column vector, and $\tilde{\sigma}(.)$ is a M by N volatility matrix, $\tilde{\sigma}^i(.)$ being its i-th line vector. We assume that $\tilde{b}(.)$ and $\tilde{\sigma}(.)$ are $\mathbb{F}$ progressively measurable processes and that $\tilde{\sigma}\tilde{\sigma}^{\mathbf{T}}(t, \omega)$ is an invertible adapted process. We also assume the existence of an $\mathbb{F}$ progressively measurable risk premium process $\tilde{\theta}(.)$, a column vector of dimension N such that for all $0 \leq t \leq T$ it satisfies:

$$\tilde{\theta}_t = \tilde{\sigma}^{\mathbf{T}}_t (\tilde{\sigma}_t \tilde{\sigma}^{\mathbf{T}}_t)^{-1} (\tilde{b}_t - r_t \mathbf{1}_M),$$

where $^{\mathbf{T}}$ denotes the transpose of a matrix and $\mathbf{1}_M$ is the M dimensional vector, where all components are equal to one. Thus $\tilde{\theta}_t$ is in the image of $\tilde{\sigma}^{\mathbf{T}}_t$. Integrability conditions on $\tilde{b}(.)$, $\tilde{\sigma}(.)$, $r(.)$ and $\tilde{\theta}(.)$ given in Chapter 3 are assumed to be satisfied.

Consider a representative agent starting from an initial wealth x. We call $\pi^i_t, i = 1, \ldots, M$ the fraction of his wealth invested at time t in each of the risky assets. For each date $0 \leq t \leq T$, the column vector of size M, $\pi_t = (\pi^1_t, \ldots, \pi^M_t)^{\mathbf{T}}$ is the portfolio process. The process π_t is assumed to be $\mathbb{F}$-progressively measurable, and such that $\sum_{j=1}^N \int_0^T ((\tilde{\sigma}^{\mathbf{T}}_t \pi_t)^j)^2 dt < +\infty$. The representative agent consumes a certain part of his wealth while he invests in the financial market, with a consumption rate c_t at time t. The consumption process is assumed to be $\mathbb{F}$-progressively measurable and such that $\int_0^T c_s ds < +\infty$. For an agent starting from an initial wealth x and taking a portfolio process π and a consumption process c, his wealth at time t is given by $X^{x,c,\pi}_t$. We are considering self-financing portfolios. We assume that we can substract a certain rate of consumption from these self-financing portfolios, at each time t. The wealth $X^{x,c,\pi}_t$ of the agent at time t is then a solution of:

$$dX^{x,c,\pi}_t = -c_t dt + X^{x,c,\pi}_t r_t dt + X^{x,c,\pi}_t \langle \tilde{\sigma}_t \pi_t, dW_t + \tilde{\theta}_t dt \rangle, \text{ and } X^{x,c,\pi}_0 = x.$$

We put in evidence here the important role of the volatility vector $\kappa_t = \tilde{\sigma}_t^{\mathbf{T}} \pi_t \in \mathbb{R}^N$. For the sake of simplicity, from now on we denote the wealth process by $X^{x,c,\kappa}(.)$, with dynamics:

$$dX_t^{x,c,\kappa} = -c_t dt + X_t^{x,c,\kappa} r_t dt + X_t^{x,c,\kappa} \langle \kappa_t, dW_t + \tilde{\theta}_t dt \rangle, \text{ and } X_0^{x,c,\kappa} = x. \quad (4.1.1)$$

The term $-c_t dt$ here is brought by consumption. The main thing to notice here is the difference of status between c_t and X_t, c_t is an instantaneous consumption rate, and X_t is an aggregate wealth see the following equation, for all $0 \le t \le T$:

$$\frac{X_t^{x,c,\kappa}}{S_t^0} = x - \int_0^t \frac{c_s}{S_s^0} ds + \int_0^t \frac{X_s^{x,c,\kappa}}{S_s^0} \langle \kappa_t, dW_t + \tilde{\theta}_t dt \rangle, \quad (4.1.2)$$

where $S_t^0 = \exp(\int_0^t r_s ds)$. Hence we see in this expression that it is the integral $\int_0^t c_s ds$ is homogeneous to a quantity of wealth, not c_t.
Throughout this chapter, we consider only wealth processes which are positive $\mathbb{P}$-almost surely for all $t \ge 0$. We say that (c, κ) are admissible and denote $(c, \kappa) \in \mathcal{A}(x)$. Other assumptions and notation relative to the description of an incomplete market (where the incompleteness comes from portfolio constraints) from Chapter 3 still hold. In particular, we consider the range of $\sigma_t^{\mathbf{T}}$. We call $\mathcal{K}_t$ the family of subvector of $\mathbb{R}^n$, such that: $\mathcal{K}_t = \tilde{\sigma}_t^{\mathbf{T}}(\mathbb{R}^M)$. For all $0 \le t \le T$, by definition $\tilde{\theta}_t$ and $\kappa_t \in \mathcal{K}_t$, and we denote $\tilde{\theta}(.) \in \mathcal{K}$, $\kappa(.) \in \mathcal{K}$.

A process $H(.)$ is a state density process, if the following process is a $\mathbb{P}$-local martingale:

$$H_t X_t^{x,c,\kappa} + \int_0^t H_s c_s ds.$$

Then there exists a progressively measurable process $\nu(.) \in \mathcal{K}^\perp$ (the orthogonal of $\mathcal{K}$ in $\mathbb{R}^N$) and the state price density process $H^\nu(.)$ is solution of the following stochastic differential equation, for all $0 \le t \le T$:

$$dH_t^\nu = -r_t dt - \langle \tilde{\theta}_t + \nu_t, dW_t \rangle, \text{ and } H_0^\nu = 1, \ \nu \in \mathcal{K}_t^\perp.$$

where $\tilde{\theta}(.)$ and $\nu(.)$ are orthogonal. As in Definition 3.1.1, we define the process $Y^{y,\nu}(.)$ such that for all $\nu \in \mathcal{K}^\perp$ and for all $y > 0$:

$$Y_t^{y,\nu} = yH_t^\nu = y\exp(-\int_0^t r_s ds - \int_0^t \langle \tilde{\theta}_s + \nu_s, dW_s \rangle - \frac{1}{2}\int_0^t ||\tilde{\theta}_s + \nu_s||^2 ds),$$

and $Y_0^{y,\nu} = y$. This process appears naturally in the expression of the dual optimization problem. It is important to remark that $y \to Y_t^{y,\nu}$ is a linear function of y.

We assume that the risk aversion of the representative agent is characterized by its preference structure (U^1, U^2), where U^1 and U^2 are continuous and twice differentiable functions on $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. The function $U^1(t, c)$ is the utility from consumption, and the function $U^2(T, x)$ is the utility from terminal wealth.
We start by recalling the (primal) utility maximization problem from consumption and terminal wealth between dates 0 and T in an incomplete market. For an agent with consumption rate $c(.)$, portfolio $\kappa(.)$ and preference structure (U^1, U^2), this problem is expressed as:

$$\sup_{(c,\kappa)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T U^1(t, c_t)dt + U^2(T, X_T^{x,c,\kappa})\right], \tag{4.1.3}$$

where (c, κ) are $\mathbb{F}$ progressively measurable processes.

4.1.2 Motivations

In the equation above (4.1.3), we see the different status of the term with the utility from consumption U^1 and the term with the utility from terminal wealth U^2. The wealth process $X_t^{x,c,\kappa}$ is an aggregate wealth, and takes the form of an integral (4.1.2). But on the other hand the consumption process c_t is a **rate of consumption**. And similarly, the utility function U^1 and U^2 have different status. U^1 is the utility of a rate of consumption and U^2 is the utility of an aggregate wealth.
Very often the same functions are used for utility fron consumption and utility from terminal wealth. But how is it possible to compare these two quantities? Is it possible to treat the consumption rate as a certain quantity of wealth? These questions are our purpose in this Chapter.

We interpret the consumption term in the following way. The process c_t is no longer considered as the consumption of the representative agent. It is now considered as an accumulation of supplies which the representative agent uses in case of an unpredictable event. Thus the agent maximizes his utility from terminal wealth or if this event happens, he maximizes the expected utility of his supplies.
Thus we treat the consumption rate as a certain quantity of wealth, as a certain quantity of supplies. These supplies are kept aside in order to face an unpredictable event. In the case where this event happens, the agent uses these supplies.
For this purpose it is necessary to introduce a new quantity in the market,

that is the random date at which the supplies are used. This unpredictable event happpening at a random date will also be called default, like in the credit risk framework.
More generally, a first naive idea is to model this random date as a random variable ζ, which is not in the market. An exponential random variable is a good example, because it is memoryless.
This random date ζ can also be interpreted as a default time.
Consider now the agent starting from an initial wealth $x > 0$, investing a part of his wealth and putting the remaining part aside to face an unpredictable event. We call $\check{X}^{x,c,\kappa}(.)$ his wealth. If the event does not happen before horizon date T, the portfolio is managed in the usual financial way. At time T, the agent maximizes his utility from terminal wealth. Let us call $\check{U}^2$ this utility function. Thus at time T the agent seeks to maximize the expectation of $\check{U}^2(\check{X}_T^{x,c,\kappa})$.
On the other hand, if the event happens before date T, then the agent uses his supplies. Let us call $\tilde{X}_t$ the value of the supplies put aside at date t. The satisfaction that the agent has when using these supplies is characterized by a utility function denoted by $\check{U}^1$. Thus, in case of an unpredictable event at date ζ, the agent maximizes the expectation of $\check{U}^1(\tilde{X}_\zeta)$.
We see here that $\check{U}^2(\check{X}_T^{x,c,\kappa})$ and $\check{U}^1(\tilde{X}_\zeta)$ have the same status: they are both utility functions of certain quantities of wealth.

To summarize this, the representative agent seeks to maximize:

$$\mathbb{E}[\check{U}^2(\check{X}_T^{x,c,\kappa})\mathbf{1}_{T<\zeta} + \check{U}^1(\tilde{X}_\zeta)\mathbf{1}_{\zeta\leq T}].$$

In this Chapter, we develop this approach.

4.1.3 Random variable ζ and filtration $\mathbb{G}$

Information on the prices of the assets of the financial market is contained in the filtration $\mathbb{F}$. We model the date at which the supplies are used as a random variable ζ, which is not in the $\mathbb{F}$-market. The global information, that is the information of $\mathbb{F}$ and of the random variable ζ is contained in a new filtration $\mathbb{G}$.
The motivation of this section is to introduce and describe the filtration $\mathbb{G}$. First we recall some definitions and properties which will be useful in the following. For more details, one can refer to Jacod [Jac79], Dellacherie and Meyer [DM75].

Hypothesis 4.1.1 *Throughout this chapter we assume that the (H) hypothesis holds.*

Let $0 < \zeta < \infty$ be a positive random variable. Then let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the minimal filtration which makes τ a $\mathbb{D}$-stopping time, i.e. $\mathcal{D}_t = \mathcal{D}^0_{t+}$ with $\mathcal{D}^0_t = \sigma(\zeta \wedge t)$. Then we consider the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ such that:

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t,$$

and assuming that $\mathcal{G}_0$ contains all null sets of $\mathcal{G}_\infty$. The filtration $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$. This filtration is often used in credit risk to model the global information in the market (the information contained in $\mathbb{F}$ and ζ), for instance see [Jea, EJY00].
Moreover, any $\mathcal{G}_t$ measurable random variable Ψ^G_t may be represented as:

$$\Psi^G_t = \Psi^F_t \mathbf{1}_{t<\zeta} + \Psi_t(\zeta)\mathbf{1}_{\zeta\leq t},$$

where Ψ^F_t is an $\mathcal{F}_t$ measurable random variable and $\Psi_t(\zeta)$ is $\mathcal{F}_t \otimes \sigma(\zeta)$ measurable.
The $\mathbb{F}$ survival processes is : $\mathbb{P}(\zeta > t|\mathcal{F}_t)$. Because the (H) hypothesis is satisfied, we have:

$$\mathbb{P}(\zeta > t|\mathcal{F}_\infty) = \mathbb{P}(\zeta > t|\mathcal{F}_t)$$

We define the process Φ_t in terms of the conditional distribution of ζ given $\mathcal{F}_\infty$:

$$\mathbb{P}[\zeta > t|\mathcal{F}_\infty] = e^{-\Phi_t}.$$

The process Φ_t is assumed to be continuous, increasing, $\mathbb{F}$ adapted and such that $\Phi_\infty = \infty$. In addition to that we assume here that Φ_t is differentiable relatively to t almost surely and we denote, for all $0 \leq t \leq T$:

$$\Phi_{t,T} = \Phi_T - \Phi_t = \int_t^T \varphi_s ds.$$

We assume that the intensity φ_t is strictly positive almost surely for all $t > 0$. Indeed, in the following paragraphs the term $\frac{1}{\varphi_t}$ appears and it has to be well defined.
In the following our purpose is to express conditional expectations relatively to the filtration $\mathbb{G}$ in terms of conditional expectations relatively to the filtration $\mathbb{F}$. For this purpose, we use the following proposition appearing in [Jea].

Proposition 4.1.1 *For a $\mathcal{F}_T$-measurable random variable X, and any $t \leq T$:*

$$\mathbb{E}[X\mathbf{1}_{T<\zeta}|\mathcal{G}_t] = \mathbf{1}_{t<\zeta}\frac{\mathbb{E}[X\mathbf{1}_{T<\zeta}|\mathcal{F}_t]}{\mathbb{E}[\mathbf{1}_{t<\zeta}|\mathcal{F}_t]} = \mathbf{1}_{t<\zeta}\mathbb{E}[Xe^{-\Phi_{t,T}}|\mathcal{F}_t]. \tag{4.1.4}$$

For a $\mathbb{F}$-predictable process Z, for every $t \leq T$:

$$\mathbf{1}_{t<\zeta}\mathbb{E}[Z_\zeta\mathbf{1}_{\zeta\leq T}|\mathcal{G}_t] = \mathbf{1}_{t<\zeta}\mathbb{E}\left[\int_t^T Z_s e^{-\Phi_{t,s}} d\Phi_s|\mathcal{F}_t\right]. \tag{4.1.5}$$

4.1.4 $\mathbb{F}$-Wealth process in the $\mathbb{G}$-market

The purpose of this section is construct an admissible wealth process in the $\mathbb{G}$-market, starting from a wealth process in the $\mathbb{F}$-market. This is a way to give some intuition about the $\mathbb{G}$-market.
Consider now a representive agent starting from an initial consumption x and taking a portfolio $\kappa(.)$ and consumption $c(.)$. We define the $\mathbb{F}$-progressively measurable process $\check{X}^{x,c,\kappa}(.)$ (mentioned in Section 4.1.2), for all $0 \leq t \leq T$ as:

$$\check{X}_t^{x,c,\kappa} = X_t^{x,c,\kappa} e^{\Phi_t},$$

where $X_t^{x,c,\kappa}$ is given by equation (4.1.1). Thus for $(c,\kappa) \in \mathcal{A}(x)$, by definition $\check{X}^{x,c,\kappa}(.)$ is positive $\mathbb{P}$-almost surely. And for all $u \geq 0$, we define the $\mathbb{F}$ progressively measurable process $\check{c}(.)$ as:

$$\check{c}_u = c_u e^{\Phi_u} (\varphi_u)^{-1}.$$

Let us now interpret $\check{c}(.)$. The consumption rate c_t is multiplied by the term $e^{\Phi_t}(\varphi_t)^{-1}$. This corresponds to the capitalization of a quantity of wealth which will be used in the future only. At each date t, a certain quantity of wealth is put in reserve (in case the event ζ occurrs before T).
Itô formula gives the dynamics of $\check{X}^{x,c,\kappa}(.)$, using (4.1.1):

$$d\check{X}_t^{x,c,\kappa} = -\check{c}_t d\Phi_t + \check{X}_t^{x,c,\kappa}(\varphi_t + r_t)dt + \check{X}_t^{x,c,\kappa}\langle \kappa_t, dW_t + \tilde{\theta}_t dt\rangle.$$

We have now the building blocks to construct a $\mathbb{G}$ progressively measurable process:

Definition 4.1.1 *We consider a representative agent with initial wealth x, portfolio $\kappa(.)$ and consumption $c(.)$ and $X^{x,c,\kappa}(.)$ the associated wealth process. Then we define the process $X^{\mathbb{G},x,\kappa}(.)$ as:*

$$\begin{aligned} X_t^{\mathbb{G},x,\kappa} &= \check{X}_t^{x,c,\kappa} \textit{ for } t < \zeta \\ X_\zeta^{\mathbb{G},x,\kappa} &= \check{c}_\zeta. \end{aligned}$$

The process $X^{\mathbb{G},x,\kappa}(.)$ defined here has one jump at time ζ.

With this formulation, we interpret the consumption rate as a certain quantity of cash. For $0 \leq t < \zeta$, the portfolio is managed in the classical financial way. Then at random date ζ, the investor uses his supplies in cash.
The expression $\check{c}_\zeta$ appears only on the event $\{\zeta \leq T\}$. Let us comment here on the integrability of $\check{c}_\zeta \mathbf{1}_{\zeta \leq T}$.

$$\mathbb{E}[\check{c}_\zeta \mathbf{1}_{\zeta \leq T}] = \mathbb{E}[c_\zeta \frac{e^{\Phi_\zeta}}{\varphi_\zeta} \mathbf{1}_{\zeta \leq T}] = \mathbb{E}[\int_0^T c_t dt] < +\infty,$$

by assumption on the integrability of the consumption process $c(.)$ in the $\mathbb{F}$-market.
It is important to notice that because the consumption rate is by definition positive almost surely, and so is $\check{X}^{x,c,\kappa}(.)$, for all $\mathbb{F}$ progressively measurable admissible strategies $(c,\kappa) \in \mathcal{A}(x)$, for all $t \geq 0$, we have:

$$X_t^{\mathbb{G},x,\kappa} \geq 0, \mathbb{P} - a.s.$$

Let us examine further the dynamics of the process $X^{\mathbb{G},x,\kappa}(.)$. Using the definition above, we have:

$$X_t^{\mathbb{G},x,\kappa} = X_t^{x,c,\kappa} e^{\Phi_t} \mathbf{1}_{t<\zeta} + X_\zeta^{\mathbb{G},x,\kappa} \mathbf{1}_{\zeta \leq t} \tag{4.1.6}$$

In this expression, $L_t^{\mathbb{G}} := e^{\Phi_t} \mathbf{1}_{t<\zeta}$ is a $\mathbb{G}$ martingale with only one jump at time ζ.
Equation (4.1.6) can be rewritten as:

$$X_t^{\mathbb{G},x,\kappa} = \check{X}_{t\wedge\zeta}^{x,c,\kappa} - (\check{X}_\zeta^{x,c,\kappa} - \check{c}_\zeta)\mathbf{1}_{\zeta \leq t} = \check{X}_{t\wedge\zeta}^{x,c,\kappa} - (X_{\zeta^-}^{\mathbb{G},x,\kappa} - X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta \leq t}, \tag{4.1.7}$$

and we denote by $\Delta X^{\mathbb{G}} := (X_{\zeta^-}^{\mathbb{G},x,\kappa} - X_\zeta^{\mathbb{G},x,\kappa})$, the jump of the $\mathbb{G}$-wealth process at time ζ. This kind of decomposition appears frequently, see for in instance [EJJ10] in the context of credit. Using the fact that $\check{X}_t^{x,c,\kappa} - \int_0^t \check{c}_t \varphi_t dt \geq 0$ a.s., we see that the jump at time ζ is negative: $X_{\zeta^-}^{\mathbb{G},x,\kappa} \geq X_\zeta^{\mathbb{G},x,\kappa}$ almost surely. That is, the agent has lost a part of his wealth at time ζ. More precisely, he has lost all his portfolio invested in the assets (with value $\check{X}_\zeta^{x,c,\kappa} = X_{\zeta^-}^{\mathbb{G},x,\kappa}$ just before default) but he has his supplies instead. The portfolio $X_t^{\mathbb{G},x,\kappa}$ is self-financing until date ζ^- only.
Let us give here a few more properties. The process

$$N_t := \mathbf{1}_{\zeta \leq t} - \int_0^{t\wedge\zeta} \varphi_s ds,$$

is a $\mathbb{G}$-martingale (see [JR00]). Its dynamics is: $dN_t = \frac{dL_t^{\mathbb{G}}}{L_{t^-}^{\mathbb{G}}}$.
For any bounded $\mathbb{G}$-predictable process $Z(.)$, the following process is a $\mathbb{G}$ martingale:

$$\int_0^t Z_u dN_u = Z_\zeta \mathbf{1}_{\zeta \leq t} - \int_0^{t\wedge\zeta} Z_u \varphi_u du. \tag{4.1.8}$$

In particular this allows to rewrite the differentials of $\check{c}_\zeta \mathbf{1}_{\zeta \leq t}$ and $\check{X}_\zeta^{x,c,\kappa}\mathbf{1}_{\zeta\leq t}$. Using the property above (4.1.8), we write the dynamics of $X_t^{\mathbb{G},x,\kappa}$:

$$
\begin{aligned}
& dX_t^{\mathbb{G},x,\kappa} \\
& = L_t^{\mathbb{G}} dX_t^{x,c,\kappa} + X_t^{x,c,\kappa}\mathbf{1}_{t<\zeta} e^{\Phi_t}\varphi_t dt - d\left((\check{X}_\zeta^{x,c,\kappa} - \check{c}_\zeta)\mathbf{1}_{\zeta\leq t}\right) \\
& = L_t^{\mathbb{G}} dX_t^{x,c,\kappa} + X_t^{x,c,\kappa}\mathbf{1}_{t<\zeta} e^{\Phi_t}\varphi_t dt - (X_{t^-}^{\mathbb{G},x,\kappa} - X_t^{\mathbb{G},x,\kappa})dN_t \\
& + (\check{c}_t - \check{X}_t^{x,c,\kappa})\mathbf{1}_{t<\zeta}\varphi_t dt \\
& = L_t^{\mathbb{G}} dX_t^{x,c,\kappa} + c_t e^{\Phi_t}\mathbf{1}_{t<\zeta} dt - (X_{t^-}^{\mathbb{G},x,\kappa} - X_t^{\mathbb{G},x,\kappa})\frac{dL_t^{\mathbb{G}}}{L_{t^-}^{\mathbb{G}}} \\
& = X_t^{x,c,\kappa}(r_t dt + \langle \kappa_t, dW_t + \tilde{\theta}_t\rangle)e^{\Phi_t}\mathbf{1}_{t<\zeta} - (X_{t^-}^{\mathbb{G},x,\kappa} - X_t^{\mathbb{G},x,\kappa})\frac{dL_t^{\mathbb{G}}}{L_{t^-}^{\mathbb{G}}},
\end{aligned}
$$

using the dynamics of dN_t.
$X_t^{\mathbb{G}}$ would be an investment strategy if it is possible to hedge it with assets. The first term of equation above takes into account the fact that the representative agent invests a certain part of his wealth in the basic assets of the market. The second term correspond to an investment in an asset with price $L_t^{\mathbb{G}}$.
We assume the existence in the $\mathbb{G}$-market of an asset, with price $L_t^{\mathbb{G}}$ at time t, which price dynamics has a jump at ζ. This asset, with price $L_t^{\mathbb{G}} = e^{\Phi_t}\mathbf{1}_{t<\zeta}$, behaves similarly to a Credit Default Swap. Assuming the existence of such an asset, it is possible to construct hedging strategies containing this asset. This assumption “completes” the market: if the initial market is complete it is possible to find a hedging strategy for $X_t^{\mathbb{G}}$. Otherwise, there is no more incompleteness due to the random variable ζ, the incompleteness may only be due to portfolio constraints. Then $X_t^{\mathbb{G},x,\kappa}$ is an admissible investment strategy in the $\mathbb{G}$-market.

4.1.5 Maximization of expected utility (primal problem)

In this section we replace the expected utility from consumption and terminal wealth maximization problem by an expected utility from terminal wealth with random horizon maximization problem. For this purpose, we define first some new functions:
Given a preference structure (U^1, U^2), we define the function $\check{U}^2(t,x)$, for all $0 \leq t \leq T$ as:

$$\check{U}^2(t,x) = U^2(t, xe^{-\Phi_t})e^{\Phi_t}.$$

And for all $0 \leq t \leq T$, we define $\check{U}^1(t,c)$ as:

$$\check{U}^1(t,c) = U^1(t, ce^{-\Phi_t})e^{\Phi_t}(\varphi_t)^{-1}.$$

Let us notice that similarly to U^1 and U^2, the functions $\check{U}^1$ and $\check{U}^2$ are assumed to be continuous on $\mathbb{R}^+ \times \mathbb{R}^+$.
The reason in the difference between $\check{U}^1$ and $\check{U}^2$ comes from the different status of U^1 and U^2 in the maximization problem. The utility from consumption function U^1 involves a rate of consumption and its integral is taken in the maximization problem, whereas only the terminal value of U^2 is considered. This difference has to be taken into account for the definition of $\check{U}^1$ and $\check{U}^2$:

Definition 4.1.2 *Let us consider a representative agent with a preference structure (U^1, U^2). Let $0 < \zeta < \infty$ be a random variable. Using the previous definition of $\check{U}^1$ and $\check{U}^2$, we define $U^{\mathbb{G}}(t,x)$ such that for all $y > 0$:*

$$U^{\mathbb{G}}(t,x) = \check{U}^2(t,x)\mathbf{1}_{t<\zeta} + \check{U}^1(\zeta,x)\mathbf{1}_{\zeta\leq t} \ a.s.$$

We can check that for a fixed $t > 0$, $U^{\mathbb{G}}(t,.)$ is an increasing concave function. Also we see that the utility function $U^{\mathbb{G}}(.)$ has one jump at ζ (whereas U^1 and U^2 are continuous functions). With this definition, we introduce a random horizon at ζ.

In this definition, the random variable ζ appears as an exogenous random vriable, which is used to introduce a random horizon.
Let us notice that $U^{\mathbb{G}}$ is $\mathbb{G}$ measurable. We have introduced an example of stochastic utility function here.

We denote by $\mathcal{A}^{\mathcal{G}}(x)$ the admissible strategies in the $\mathbb{G}$-market, starting from an initial wealth x. That is, we say that $\kappa \in \mathcal{A}^{\mathcal{G}}(x)$ if $X_t^{x,\kappa}$ is positive $\mathbb{P}$-almost surely, for all $t \geq 0$.

Definition 4.1.3 *Between dates 0 and T, the representative agent maximizes the expected utility of his terminal wealth if the event ζ does not happen before T. Otherwise he maximizes the expected utility of his supplies at default date ζ. That is:*

$$\sup_{\kappa\in\mathcal{A}^{\mathcal{G}}(x)} \mathbb{E}\left[U^{\mathbb{G}}(T, X_T^{\mathbb{G},x,\kappa})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta\leq T}\right]. \tag{4.1.9}$$

This is the maximization problem in the $\mathbb{G}$-market.

Here we show that the maximization problem in the $\mathbb{G}$-market is in fact another way to rewrite the consumption/investment problem in the $\mathbb{F}$-market. The expression above (4.1.9) is equivalent to:

$$\mathbb{E}\left[U^{\mathbb{G}}(T, X_T^{\mathbb{G},x,\kappa})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta\leq T}\right] = \mathbb{E}\left[\check{U}^2(T, \check{X}_T)\mathbf{1}_{T<\zeta} + \check{U}^1(\zeta, \check{c}_\zeta)\mathbf{1}_{\zeta\leq T}\right]$$
$$= \mathbb{E}\left[U^2(T, X_T^{x,c,\kappa})e^{\Phi_T}\mathbf{1}_{T<\zeta} + U^1(\zeta, c_\zeta)\frac{e^{\Phi_T}}{\varphi_T}\mathbf{1}_{\zeta\leq T}\right]$$
$$= \mathbb{E}\left[U^2(T, X_T^{x,c,\kappa})\right] + \mathbb{E}\left[\int_0^T U^1(z, c_z)dz\right].$$

On the left hand side of this expression, we have an expected utility from terminal wealth in the $\mathbb{G}$ market. One can check that the consumption rate no longer appears explicitly. We see that by using the random variable ζ, we have introduced a random horizon. This is the expression to maximize in the $\mathbb{G}$-market between dates 0 and T.
On the right hand side of this expression, there are two terms, one involving the utility from consumption and one involving the utility from terminal wealth, this is the usual expected utility from consumption and terminal wealth in the $\mathbb{F}$-market.

We see that the maximization of expected utility from consumption problem can be seen as a maximization of expected utility from terminal wealth problem, in a framework where the wealth process as been replaced by $X^{\mathbb{G},x,\kappa}(.)$.
Example. We examine a simple example of random variable ζ. If ζ is an exponential random variable with parameter $\varphi > 0$, his survival process is given by $e^{-\varphi t}$ and $\Phi_t = \varphi t$, for all $t \geq 0$. We obtain:

$$\mathbb{E}\left[U^{\mathbb{G}}(T, X_T^{\mathbb{G},x,\kappa})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta\leq T}\right]$$
$$= \mathbb{E}\left[U^2(T, X_T^{x,c,\kappa})e^{-\varphi T}\mathbf{1}_{T<\zeta} + U^1(\zeta, c_\zeta)\frac{e^{-\varphi T}}{\varphi\, T}\mathbf{1}_{\zeta\leq T}\right]$$
$$= \mathbb{E}\left[U^2(T, X_T^{x,c,\kappa})\right] + \mathbb{E}\left[\int_0^T U^1(z, c_z)dz\right]$$

■

Now we also show how this expression is modified in the case of a maximization problem between dates t and T.

Remark 4.1.1 *For the conditional problem, the expression to maximize in* *the $\mathbb{G}$-market is:*

$$\sup_{\kappa\in\mathcal{A}^{\mathcal{G}}(x)} \mathbb{E}[U^{\mathbb{G}}(T, X_T^{\mathbb{G}})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le T}|\mathcal{G}_t]$$
$$= \sup_{(c,\kappa)\in\mathcal{A}(x)} \mathbb{E}[U^2(T, X_T^{x,c,\kappa}) + \int_t^T U^1(s, c_s)ds|\mathcal{F}_t]e^{\Phi_t}\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le t}.$$

Proof. One can check that, for all $0 \le t \le T$:

$$\mathbb{E}[U^{\mathbb{G}}(T, X_T^{\mathbb{G}})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le T}|\mathcal{G}_t] = \mathbb{E}[\check{U}^2(T, X_T^{\mathbb{G}})\mathbf{1}_{T<\zeta} + \check{U}^1(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le T}|\mathcal{G}_t]$$
$$= \mathbb{E}[\check{U}^2(T, X_T^{\mathbb{G}})\mathbf{1}_{T<\zeta} + \check{U}^1(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{t<\zeta\le T} + \check{U}^1(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le t}|\mathcal{G}_t]$$
$$= \mathbb{E}[U^{\mathbb{G}}(T, X_{T\wedge\zeta}^{\mathbb{G}})|\mathcal{G}_t]\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta\le t}.$$

Then we rewrite the first term in the expression above.
Applying equation (4.1.4) and (4.1.5) in our case gives the expectation of $U^{\mathbb{G}}(T, X_{T\wedge\zeta}^{\mathbb{G},x,\kappa})$ relatively to the filtration $\mathbb{G}$ on $\{t < \zeta\}$:

$$\mathbb{E}[U^{\mathbb{G}}(T, X_{T\wedge\zeta}^{\mathbb{G},x,\kappa})|\mathcal{G}_t]\mathbf{1}_{t<\zeta} = (\mathbb{E}[\check{U}^2(T, \check{X}_T^{x,c,\kappa})\mathbf{1}_{T<\zeta}|\mathcal{G}_t] + \mathbb{E}[\check{U}^1(\zeta, \check{c}_\zeta)\mathbf{1}_{\zeta\le T}|\mathcal{G}_t])\mathbf{1}_{t<\zeta}$$
$$= \big(\mathbb{E}[\check{U}^2(T, \check{X}_T^{x,c,\kappa})e^{-\Phi_{t,T}}|\mathcal{F}_t] + \mathbb{E}\big[\int_t^T \check{U}^1(s, \check{c}_s)e^{-\Phi_{t,s}}d\Phi_s|\mathcal{F}_t\big]\big)\mathbf{1}_{t<\zeta}.$$
$$= \big(\mathbb{E}[U^2(T, X_T^{x,c,\kappa})|\mathcal{F}_t] + \mathbb{E}\big[\int_t^T U^1(s, c_s)ds|\mathcal{F}_t\big]\big)e^{\Phi_t}\mathbf{1}_{t<\zeta}.$$

■

4.1.6 A backward formulation of the primal maximization problem

The backward point of view of the primal utility maximization problem, with ξ_T a $\mathcal{F}_T$ measurable positive random variable (attainable wealth) is:

$$\sup_{(c,\xi_T)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T U^1(t, c_t)dt + U^2(T, \xi_T)\right],$$

under the budget constraint:

$$\sup_{\nu\in\mathcal{K}^\perp} \mathbb{E}\left[\int_0^T c_t H_t^\nu dt + H_T^\nu \xi_T\right] \le x. \tag{4.1.10}$$

We assume that the optimum of the problem above (4.1.10) is attained by a so-called optimal dual process $\nu^*(.)$.

In this section, we give the counterpart of this maximization problem in the $\mathbb{G}$-market.

Let us show that $X_t^{\mathbb{G},x,\kappa} H_t^\nu$ is a $\mathbb{G}$ local martingale. For this purpose, let us notice first the following property.

Lemma 4.1.1 *Let m_t be an $\mathbb{F}$-martingale. Then $m_t L_t^{\mathbb{G}}$ is a $\mathbb{G}$ martingale. This is due to the fact that for all $0 \le t \le s$, we have:*

$$\begin{aligned}
\mathbb{E}[m_s L_s^{\mathbb{G}} | \mathcal{G}_t] &= \mathbb{E}[\boldsymbol{1}_{\zeta > s} m_s L_s^{\mathbb{G}} | \mathcal{G}_t] \\
&= \boldsymbol{1}_{\zeta > t} e^{\Phi_t} \mathbb{E}[\boldsymbol{1}_{\zeta > s} e^{\Phi_s} m_s | \mathcal{F}_t] \\
&= L_t^{\mathbb{G}} \mathbb{E}[\mathbb{E}[\boldsymbol{1}_{\zeta > s} | \mathcal{F}_t] e^{\Phi_s} m_s | \mathcal{F}_t] = L_t^{\mathbb{G}} \mathbb{E}[m_s | \mathcal{F}_t].
\end{aligned}$$

For all $0 \le t \le T$, the expression $X_t^{\mathbb{G},x,\kappa} H_t^\nu$ can be rewritten as a sum of three terms:

$$\begin{aligned}
&X_t^{\mathbb{G},x,\kappa} H_t^\nu \\
&= X_t^{x,c,\kappa} H_t^\nu L_t^{\mathbb{G}} + \check{c}_\zeta H_\zeta^\nu \mathbf{1}_{\zeta \le t} \\
&= \big(X_t^{x,c,\kappa} H_t^\nu + \int_0^t c_s H_s^\nu ds \big) L_t^{\mathbb{G}} + \big(\int_0^t c_s H_s^\nu L_s^{\mathbb{G}} - \int_0^t c_s H_s^\nu L_t^{\mathbb{G}} ds \big) \\
&+ \big(\check{c}_\zeta H_\zeta^\nu \mathbf{1}_{\zeta \le t} - \int_0^t \check{c}_s H_s^\nu \varphi_s \mathbf{1}_{\zeta > s} ds \big)
\end{aligned}$$

Using (4.1.8), the last term is a $\mathbb{G}$-martingale, so is the second one. We examine now the first term. This expression is the product of $L_t^{\mathbb{G}}$ and an $\mathbb{F}$-martingale. Using the property above (4.1.1), shown in [JR00] for instance, this product is a $\mathbb{G}$ local martingale. Thus, $X_t^{\mathbb{G},x,\kappa} H_t^\nu$ is a positive $\mathbb{G}$ local martingale, thus a supermartingale. It is possible to deduce the following budget constraint:

$$\mathbb{E}[X_{T\wedge\zeta}^{\mathbb{G},x,\kappa} H_{T\wedge\zeta}^\nu | \mathcal{G}_t] \le x. \tag{4.1.11}$$

Proposition 4.1.2 *Let us define a $\mathbb{G}$ adapted process $\xi_t^{\mathbb{G}}$, such that:*

$$\xi_t^{\mathbb{G}} = \xi_t^{\mathbb{F}} \mathbf{1}_{t < \zeta} + \check{c}_\zeta \mathbf{1}_{\zeta \le t}.$$

Then the maximization problem in the $\mathbb{G}$-market between dates 0 and T is:

$$\sup_{\xi \in \mathcal{A}^{\mathcal{G}}(x)} \mathbb{E}\left[U^{\mathbb{G}}(T, \xi_T^{\mathbb{G}}) \mathbf{1}_{T < \zeta} + U^{\mathbb{G}}(\zeta, \xi_\zeta^{\mathbb{G}}) \mathbf{1}_{\zeta \le T} \right],$$

under the budget constraint:

$$\sup_{\nu \in \mathcal{K}^\perp} \mathbb{E}[\xi_{T\wedge\zeta}^{\mathbb{G}} H_{T\wedge\zeta}^\nu] \le x,$$

Proof. Any $\mathcal{G}_t$ measurable random variable can be rewritten as:

$$\xi_t^{\mathbb{G}} = \Psi_t^{\mathbb{F}} \mathbf{1}_{t<\zeta} + \Psi_t(\zeta)\mathbf{1}_{\zeta \le t},$$

thus it is possible to choose a $\mathbb{G}$ adapted process, such that for all $t \geq 0$: $\xi_t^{\mathbb{G}} = \xi_t^{\mathbb{F}} \mathbf{1}_{t<\zeta} + \check{c}_\zeta \mathbf{1}_{\zeta \le t}$, where $\xi_t^{\mathbb{F}}$ is an $\mathbb{F}$ adapted process. Then with a proof similar to the one of the previous subsection:

$$\mathbb{E}\left[U^{\mathbb{G}}(T, \xi_T^{\mathbb{G}})\mathbf{1}_{T<\zeta} + U^{\mathbb{G}}(\zeta, \xi_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta \le T}\right] = \mathbb{E}\left[U^2(T, \xi_T^{\mathbb{F}}) + \int_0^T U^1(t, c_t)dt\right]$$

And we add the budget constraint (4.1.11).
■

4.1.7 The dual problem

On the other hand we consider now the dual optimization problem in the $\mathbb{F}$-market associated with the primal problem mentioned above. Using the notations from Chapter 3, the dual problem is, for a fixed $y > 0$, to minimize over $\nu \in \mathcal{K}^\perp$ the following expression:

$$\tilde{V}(y) = \inf_{\nu \in \mathcal{K}^\perp} \mathbb{E}\left[\int_0^T \tilde{U}^1(t, Y_t^{y,\nu})dt + \tilde{U}^2(T, Y_T^{y,\nu})\right],$$

where $\tilde{V}(y)$ is the value function of the dual problem.

In this section, we write the dual problem in the $\mathbb{G}$-market.
We consider an agent with a preference structure (U^1, U^2) and we call $(\tilde{U}^1, \tilde{U}^2)$ the associated Fenchel transforms. We recall that these Fenchel transforms are defined by:

$$\tilde{U}^1(t, y) = \inf_{c>0}\{U^1(t, c) - cy\}$$

$$\tilde{U}^2(t, y) = \inf_{x>0}\{U^2(t, x) - xy\}$$

Given a preference structure (U^1, U^2), we define the function $\check{\tilde{U}}_2(t, y)$, for all $0 \le t \le T$ as:

$$\check{\tilde{U}}_2(t, y) = \tilde{U}^2(t, y)e^{\Phi_t}.$$

And for all $0 \le t \le T$, we define $\check{\tilde{U}}_1(t, y)$ as:

$$\check{\tilde{U}}_1(t, y) = \tilde{U}^1(t, y)e^{\Phi_t}(\varphi_t)^{-1}.$$

Definition 4.1.4 *Let consider a representative agent with a preference stucture (U^1, U^2). Let $\zeta > 0$ be a random variable independant from $\mathcal{F}_\infty$. Using the previous definition of $\check{\tilde{U}}_1$ and $\check{\tilde{U}}_2$, we define $\tilde{U}^{\mathbb{G}}(t,y)$ such that for all $y > 0$, for all $0 \leq t \leq T$:*

$$\tilde{U}^{\mathbb{G}}(t,y) = \check{\tilde{U}}_2(t,y)\mathbf{1}_{t<\zeta} + \check{\tilde{U}}_1(\zeta,y)\mathbf{1}_{\zeta\leq t}.$$

On can check that with this definition, we have:

$$\tilde{U}^{\mathbb{G}}(t,y) = \inf_{x>0}\{U^{\mathbb{G}}(t,x) - xy\}$$

The function $\tilde{U}^{\mathbb{G}}$ is the Fenchel transform of $U^{\mathbb{G}}$, it is the dual utility function of $U^{\mathbb{G}}$.

The difference between the two terms in equation above is that $\check{\tilde{U}}_1(\zeta, y)$ will be integrated relatively to ζ whereas $\check{\tilde{U}}_2(t,y)$ has to be defined for each date t.

We write the dual maximization problem in the $\mathbb{G}$-market as:

Proposition 4.1.3 *The dual utility maximization problem in the $\mathbb{G}$-market is:*

$$\tilde{V}(y) = \inf_{\nu\in\mathcal{K}^\perp}\mathbb{E}[\tilde{U}^{\mathbb{G}}(T, Y_T^{y,\nu})\mathbf{1}_{T<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq T}].$$

In the formulation of the dual problem in the $\mathbb{G}$-market, the process $Y^{y,\nu}(.)$ is the same as in the $\mathbb{F}$-market, only stopped at the random horizon ζ; this process has no jumps. Thus, here the new utility function $\tilde{U}^{\mathbb{G}}$ has a jump at ζ and the process $Y^{y,\nu}(.)$ has no jumps. This is the significant difference with the previous section where $\tilde{U}^{\mathbb{G}}$ and $X^{\mathbb{G}}(.)$ both had jumps at ζ. Thus, the dual maximization problem has a more simple expression, this is why we priviledge its use.

Proof. For the same reasons as in the case of the primal problem, we have:

$$\mathbb{E}[\tilde{U}^{\mathbb{G}}(T, Y_T^{y,\nu})\mathbf{1}_{T<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq T}] = \mathbb{E}[\tilde{U}^2(T, Y_T^{y,\nu}) + \int_0^T \tilde{U}^1(s, Y_s^{y,\nu})ds].$$

■

And for the conditional problem:

$$\inf_{\nu\in\mathcal{K}^\perp}\mathbb{E}[\tilde{U}^{\mathbb{G}}(T, Y_T^{y,\nu})\mathbf{1}_{T<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq T}|\mathcal{G}_t]$$
$$= \inf_{\nu\in\mathcal{K}^\perp}(\mathbb{E}[\check{\tilde{U}}_2(T, Y_T^{y,\nu})\mathbf{1}_{T<\zeta}|\mathcal{G}_t] + \mathbb{E}[\check{\tilde{U}}_1(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq T}|\mathcal{G}_t])\mathbf{1}_{t<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq t}$$
$$= \inf_{\nu\in\mathcal{K}^\perp}\mathbb{E}[\tilde{U}^2(T, Y_T^{y,\nu}) + \int_t^T \tilde{U}_1(s, Y_s^{y,\nu})ds|\mathcal{F}_t]e^{\Phi_t}\mathbf{1}_{t<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu})\mathbf{1}_{\zeta\leq t}.$$

4.1.8 Parametrization of the dual problem by initial consumption

Here we show that the initial optimal consumption c_0^* appears naturally in the expression of the dual problem. Thus the dual problem can be parametrized by the initial optimal consumption.
When an agent with initial wealth x maximizes his expected utility from consumption and terminal wealth in an $\mathbb{F}$-market, we know that $c_0^* = I_1(0, \mathcal{Y}(x))$, where $\mathcal{Y}(x)$ is the Lagrange multiplier. Because $I_1 \equiv \tilde{U}_1'$, the previous expression can also be rewritten as:

$$c_0^* = \tilde{U}_1'(0, y),$$

where y is the Lagrange multiplier, such that $y = \mathcal{X}_3(x)$. Or also:

$$y = (\tilde{U}_1')^{-1}(0, c_0^*).$$

Thus, the dual problem, where the key parameter is $\mathcal{Y}(x)H_t^{\nu^*}$, is in fact parametrized by $(\tilde{U}_1')^{-1}(0, c_0^*)H_t^{\nu^*}$, that is by the initial consumption.

This remark can be used in the $\mathbb{G}$-market to express the optimal state price density as a function of the optimal consumption. Denote by $\tilde{U}_y^{\mathbb{G}}$ the derivative of $\tilde{U}^{\mathbb{G}}$ relatively to its second variable. Let us denote by $X^{\mathbb{G},*}(.)$ the optimal wealth process in the $\mathbb{G}$-market. Calculations similar to the classical case show that: $X_{T\wedge\zeta}^{\mathbb{G},*} = \tilde{U}_y^{\mathbb{G}}(T \wedge \zeta, Y_{T\wedge\zeta}^{y,\nu^*})$. Thus at ζ, the optimal consumption is given by:

$$\check{c}_\zeta^* = \tilde{U}_y^{\mathbb{G}}(\zeta, Y_\zeta^{y,\nu^*}).$$

Thus, the optimal state price density process can be rewritten as: $Y_\zeta^{y,\nu^*} = (\tilde{U}_y^{\mathbb{G}})^{-1}(\zeta, \check{c}_\zeta^*)$.
We will also take advantage of this kind of formulation in Chapter 5 in the case of dynamic utility functions.

4.1.9 Conclusion

We conclude here with a few remark concerning the representative agent's consumption. In the work of [Gol] on long term investments, the consumption plays a crucial role. More precisely, the representative agent invests a certain fraction of his wealth for future generations and consumes entirely what is left. The representative agent tries to help future generations by limiting his own consumption so that there is more left to consume for future generations.

And when future generations receive the benefit of this investment, they consume it entirely.
The approach introduced in this Chapter is a different one. But in any case this example from [Gol] shows that interest rates and consumption are thus two key quantities in the economic point of view. The Ramsey Rule makes the link between interest rates and consumption and shows how the yield curve depends directly on consumption.
But in finance, putting the emphasis on consumption is not common. The maximization problem is parametrized by wealth (remember the fact that we always consider a representative agent with initial wealth x). Thus our main question is: how is it possible to find a new interpretation of the consumption process?
In order to answer this question, in this Chapter, we suggest a new point of view of the consumption process. We see that it is possible to consider the consumption as a certain quantity of wealth which is put aside for the case where an event happens.

The problem of maximizing utility from consumption and terminal wealth can be rewritten as a problem of maximizing an utility $U^{\mathbb{G}}$ from terminal wealth only in a new $\mathbb{G}$-market, with a random horizon.
This new function $U^{\mathbb{G}}$ is an example of stochastic utility function. This contributes to motivate the use of dynamic utility functions which are the purpose Chapter 5.

The same kind of approach from the point of view of the dual problem shows that the new dual problem in the $\mathbb{G}$-market still depends on the state price density process $Y^{y,\nu}(.)$. We priviledge this approach for its simplicity: the state price density process $Y^{y,\nu}$ is the same in the $\mathbb{G}$ market as in the $\mathbb{F}$ market, there is no need to define a new state price density process.

4.2 Ambiguity of a parameter

We have previously mentioned the fact that long term issues were characterized by a tremendous uncertainty. So we would like to take into account the uncertainty on the parameters of the economy in the long term (that is for a time horizon of 30 or 50 years at least).
Thus we integrate to our approach the case where there is ambiguity on one of the parameters of the model. This kind of situation is mentioned by Gollier [Gol09c], where an agent has to take a decision about his investment but there is an uncertainty on a parameter of the economy. Taking this kind of uncertainty into account makes especially sense for long-term issues, where it is clear that the agent cannot be certain about many parameters of the economy in the future.
The main purpose of this section is to show how the yield curve is modified when there is uncertainty on a parameter, and what becomes the Ramsey Rule. For this purpose we will use the same kind of approach as in the previous chapters. This means that we solve the maximization of expected utility from consumption problem when there is uncertainty on a parameter. Thus we obtain the optimal consumption process. Then we derive the new expression of the Ramsey Rule and of the yield curve.
For the sake of simplicity, we consider the case of the complete financial market $\mathcal{M}$ used in Chapter 2 and we study the expected utility maximization problem between 0 and T. We also assume that the terminal wealth at time T is equal to zero, thus we consider the expected utility maximization problem from consumption only (Problem 1), with value function V_1.

The ambiguity on a parameter is modelled by a random variable L. We denote by μ^L the distribution of the random variable L on $\mathbb{R}$. The ambiguity on a parameter that we introduce represents in some sense a form of incompleteness of the market. Because the framework in this section is the complete market $\mathcal{M}$ (no portfolio contraints), the only form of incompleteness comes from the introduction of this random variable L.

4.2.1 A special case of ambiguity

Assumption 2 *In this section, we assume that the random variable L is independent from the filtration $\mathbb{F}$.*

First of all, let us assume that the realization of the random variable is $L = l$. The representative agent starts from an initial wealth x_0^l and takes a

$\mathbb{F}$-progressively measurable consumption path $c^l(.)$. His risk aversion is characterized by the utility from consumption function $U^l(t,c)$ and he maximizes his expected utility from consumption. This is a classical maximization problem of expected utility from consumption in a complete market for an agent endowed with a standard utility function $U^l(t,c)$ and starting from an initial wealth x_0^l, with value function $V^l(x_0^l)$:

$$V^l(x_0^l) = \sup_{c^l \geq 0} \mathbb{E}\left[\int_0^T U^l(t, c_t^l)dt\right] \text{ s.t. } \mathbb{E}\left[\int_0^T H_t^0 c_t^l dt\right] \leq x_0^l,$$

Its solution, the optimal consumption path, is for all $0 \leq t \leq T$:

$$c_t^{l,*}(x_0^l) = I^l(t, y^l H_t^0), \tag{4.2.1}$$

where I^l is the inverse function of the derivative U_c^l and for all $l \in \mathbb{R}$, the constant y^l is determined by the budget constraint:

$$x_0^l = \mathcal{X}^l(y^l) = \mathbb{E}\left[\int_0^T I^l(t, y^l H_t^0)H_t^0 dt\right], \tag{4.2.2}$$

which can be rewritten as $y^l = \mathcal{Y}^l(x_0^l)$. The notation $c_t^{l,*}(x_0^l)$ comes from the fact that the optimal consumption path depends on x_0^l. The agent who solves this maximization problem is aware of a realization $L = l$. We call him the "conditional agent".

Later in this section, the dual formulation of this maximization problem will be useful. For all $y > 0$, the dual value function $\tilde{V}^l(y)$ is linked to the dual utility functions $\tilde{U}^l(t,y)$ by the relation:

$$\tilde{V}^l(y) = \mathbb{E}\left[\int_0^T \tilde{U}^l(t, Y_t^0(y))dt\right],$$

where $Y_t^0(y) := yH_t^0$.

Now let us describe the representative agent who has information on the distribution μ^L of L. This first definition characterizes the preference structure of this agent:

Definition 4.2.1 *The who has information on μ^L is endowed with the aggregated utility function $U(t,c)$, which takes into account all the possible outcomes $L = l$ and the distribution of L. Let us define this function:*

$$U(t,c) = \max\left\{\int_{\mathbb{R}} U^l(t,c^l)d\mu^L(l), \int_{\mathbb{R}} c^l d\mu^L(l) \leq c, c^l \geq 0, \forall l\right\}. \tag{4.2.3}$$

The function $U(t,c)$ is the sup-convolution of the functions $U^l(t,c^l)$.
It is more useful to define this function from the dual point of view because of the simplicity of the formulation:

$$\tilde{U}(t,y) = \int_{\mathbb{R}} \tilde{U}^l(t,y)d\mu^L(l).$$

At the optimum the constraint on the processes c^l is saturated, that is the maximum in equation above is achieved for an optimal family $(\hat{c}^l)_{l\in\mathbb{R}}$ such that:

$$\int_{\mathbb{R}} \hat{c}^l d\mu^L(l) = c,$$

Remark 4.2.1 *This definition reminds us of an Arrow-Debreu equilibrium (we refer for example to Dana and Jeanblanc, in Chapter 7 of [DJ98]). In an Arrow-Debreu equilibrium, the aggregated utility function $u(t,c)$ is the one maximizing the term $\sum_i \alpha_i u^i(t,x_i)$, where the x_i are positive and satisfy the constraint $\sum_i x_i \leq c$. The coefficients α_i represent the importance of each agent in the equilibrium.*
In our case, the agent who knows μ^L aggregates the utility functions U^l corresponding to each outcome $L = l$, with each weight representing the probability that $L = l$.

The agent who knows μ^L starts from an initial wealth x, which is assumed to be given. The preference structure $U(t,c)$ of this agent has been given above. We denote by c_t the consumption process of the representative agent with information on the distribution μ^L of L. At each date t, the value of $U(t,c_t)$ is given by the sup-convolution problem above.
He solves the following maximization problem:

$$V(x) = \sup_{c\geq 0} \mathbb{E}\left[\int_0^T U^1(t,c_t)dt\right] \text{ s.t. } \mathbb{E}\left[\int_0^T H_t^0 c_t dt\right] \leq x.$$

The corresponding dual value function is for all $y > 0$:

$$\tilde{V}(y) = \mathbb{E}\left[\int_0^T \tilde{U}(t,Y_t^0(y))dt\right].$$

It is then possible to show the following sup-convolution relation.

Proposition 4.2.1 *The relation between the value function of the agent with information on μ^L and the conditional agents is:*

$$V(x) = \{\sup_{z_0^l} \int_{\mathbb{R}} V^l(z_0^l)d\mu^L(l), \int_{\mathbb{R}} z_0^l d\mu^L(l) = x\}.$$

The value function $V(x)$ is known as the sup-convolution of the concave value functions V^l of the "conditional" agents. And the following relation exists between the dual value functions. For $y > 0$:

$$\tilde{V}(y) = \int_{\mathbb{R}} \tilde{V}^l(y) d\mu^L(l).$$

Proof. It is more practical to prove this property through dual value functions. Using the definition of $U(t,c)$ as a sup-convolution of the functions $U^l(t,c^l)$,the dual value function $\tilde{V}(y)$ can be rewitten as:

$$\begin{aligned}\tilde{V}(y) = \mathbb{E}\left[\int_0^T \tilde{U}(t, Y_t^0(y))dt\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \tilde{U}^l(t, Y_t^0(y)) d\mu^L(l) dt\right] \\ &= \int_{\mathbb{R}} \tilde{V}^l(y) d\mu^L(l),\end{aligned}$$

where the last step comes form the use of Fubini's theorem. ■

For this kind of relations between value functions, we refer to Karatzas [KS98], pages $116 - 117$, where he establishes sup-convolution relaton between value functions for Problems 1, 2 and 3.

Let us now notice a consequence of the formulation of the maximization problem for the conditional agents. For all $l \in \mathbb{R}$ we have:

$$U_c^l(t, c_t^{l,*}) = H_t^0 U_c^l(0, c_0^{l,*}). \tag{4.2.4}$$

The optimal consumptions $c^{l,*}(.)$ are Pareto optimal.

We mention the work of Jouini, Marin and Napp [JMN10]. They consider an equilibrium, where the different l are different agents with different beliefs. In this case each agent has his own probability distribution characterizing his beliefs, a different state price density process $H^l(.)$. Thus each agent has a different yield curve $R^l(.)$ given by the Ramsey Rule. This corresponds to the fact that different economists have different beliefs about the yield curve and the value of its parameters. An equilibrium is established, a yield curve $R(.)$ depending on the different $R^l(.)$.
In our framework, future work might explore the consequences on the yield curve when taking into account different beliefs.

4.3 Full information on L in complete markets

Another situation, which we examine here, is the case where the representative agent knows the realization of this uncertain parameter of the model.

Thus, in this section, we assume that the representative agent has full information on the random variable L.

Here, similarly to section (4.2.1) we assume that we are in the case of a complete financial market $\mathcal{M}$, such as the market described in Chapter 2 and we examine the problem of optimizing the utility from consumption only.

In this section we refer to classical utility maximization theory such as [KS98] and filtration enlargement [Jac79, Jac85, GP98] and more recent references, for instance [Hil04, HJ10].

Remark 4.3.1 *A special case would be the situation where L is independent from the Brownian filtration $\mathbb{F}$, that is for any $t \in [0, T]$:*

$$\mathbb{P}(L \in .|\mathcal{F}_t) = \mathbb{P}(L \in .),\ \forall t \geq 0, \mathbb{P} - a.s.$$

But in this section, we present the more general case where the random variable L is not independent from $\mathbb{F}$.

We define the filtration $\mathbb{G}^L = (\mathcal{G}_t^L)_{0 \leq t \leq T}$, as:

$$\mathcal{G}_t^L := \mathcal{F}_t \vee \sigma(L),$$

and assuming that $\mathcal{G}_0^L$ contains all null sets of $\mathcal{G}_\infty^L$. The filtration $\mathbb{G}^L$ is the initial enlargement of the filtration $\mathbb{F}$.

In the theory of initial enlargement of filtration, it is standard to work under the following density hypothesis due to Jacod [Jac79, Jac85].

Hypothesis 4.3.1 *We assume that the random variable L satisfies the assumption:*

$$\mathbb{P}(L \in \cdot|\mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot), \quad \forall t \geq 0,\, \mathbb{P} - a.s$$

Jacod has shown that, if Hypothesis 4.3.1 is fulfilled, then any $\mathbb{F}$-local martingale is a $\mathbb{G}^L$-semimartingale.

We denote by $P_t^L(\omega, dx)$ a regular version of the conditional law of L given $\mathcal{F}_t$ and by P^L the law of L (under the probability $\mathbb{P}$). According to [Jac85], there exists a measurable version of the conditional density

$$p_t(x)(\omega) = \frac{dP_t^L}{dP^L}(\omega, x) \tag{4.3.1}$$

which is an $(\mathbb{F}, \mathbb{P})$-martingale and hence can be written as:

$$p_t(x) = p_0(x) + \int_0^t \beta_s(x) dW_s, \quad \forall x \in \mathbb{R},$$

for some $\mathbb{F}$-predictable process $(\beta_t(x))_{t\geq 0}$ (we recall that in this last equation, $W(.)$ is the $\mathbb{P}$-Brownian motion previously defined.
If L is independent from $\mathbb{F}$, then $p \equiv 1$. The fact that P_t^L is equivalent to P^L implies that $\mathbb{P}$-almost surely $p_t(L) > 0$. Let us introduce the $\mathbb{F}$-predictable process ρ^L (information drift) where $\rho_t^L(x) = \beta_t(x)/p_t(x)$, the density process $p_t(L)$ satisfies the following stochastic differential equation:

$$dp_t(L) = p_t(L)\rho_t^L(L)dW_t.$$

Then $\widetilde{B}_t^L := W_t - \int_0^t \rho_s^L(L)ds, t \geq 0$, is a $(\mathbb{G}^L, \mathbb{P})$-Brownian motion.

It has been proved in [GP98] that the Hypothesis 4.3.1 is satisfied if and only if there exists a probability measure equivalent to $\mathbb{P}$ and under which $\mathcal{F}_\infty := \cup_{t\geq 0}\mathcal{F}_t$ and $\sigma(L)$ are independent (and we recall the assumption that $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$). The probability $\mathbb{P}^L$ defined by the density process

$$\mathbb{E}_{\mathbb{P}^L}\Big[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^L}\big|\,\mathcal{G}_t^L\Big] = p_t(L)$$

is the only one satisfying this condition and is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$.

We introduce the process Y^L such that:

$$Y_t^L := \mathcal{E}\bigg(-\int_0^t \rho_s^L(L)d\widetilde{B}_s^L\bigg) = \exp\bigg(-\int_0^t \rho_s^L(L)d\widetilde{B}_s^L - \frac{1}{2}\int_0^t (\rho_s^L(L))^2 ds\bigg), \tag{4.3.2}$$

where we recall that $\mathcal{E}$ is the Doléans-Dade exponential. We have:

$$d((Y_t^L)^{-1}) = (Y_t^L)^{-1}\rho_t^L(L)dW_t.$$

Thus, $Y_t^L = \frac{1}{p_t(L)}$, that is, Y_t^L is the Radon-Nikodym density of the change of probability $\mathbb{P}^L$ with respect to $\mathbb{P}$ on $\mathcal{G}_t^L$. The process Y^L is an $\mathbb{F}$-adapted process which is an $\mathbb{F}$-local martingale under the $\mathbb{F}$ risk-neutral probability $\mathbb{Q}$ (which is equivalent to $\mathbb{P}$).
We define a new probability measure $\mathbb{Q}^L$ by:

$$\mathrm{d}\mathbb{Q}^L = Y_t^L \mathrm{d}\mathbb{Q} \quad \text{on } \mathcal{G}_t^L.$$

then any $(\mathbb{F}, \mathbb{Q})$-local martingale is an $(\mathbb{G}^L, \mathbb{Q}^L)$-local martingale.
From this comes the existence of the probability $\mathbb{Q}^L$, the risk-neutral probability for the filtration $\mathcal{G}^L$. Moreover we have $\mathbb{Q}^L \simeq \mathbb{P}$ and we denote by $Z^L(.)$ the density:

$$\frac{d\mathbb{Q}^L}{d\mathbb{P}} = Z_t^L,$$

that is:

$$Z_t^L = \mathcal{E}\big(-\int_0^t \langle \theta_s + \rho_s^L(L), \widetilde{B}_s^L\rangle\big), \text{ and } Z_0^L = 1.$$

This change of probability density contains the risk premium process and the information drift. For all $0 \le t \le T$, we define the state density price process $H_t^L = Z_t^L / S_t^0$. That is:

$$H_t^L = H_t^0 Y_t^L.$$

If L is independent from $\mathbb{F}$, then $p \equiv 1$ and $H^L(.) \equiv H^0(.)$.

Because L is a random variable, there is an uncertainty on the consumption path. We denote by c_t^L the consumption process of the representative agent, and X_t^L is the corresponding wealth process. The representative agent who has full information on the random variable L solves the maximization problem (with x_0^L a $\sigma(L)$-measurable random variable):

$$V^L(x_0^L) = \sup_{c^L \ge 0} \mathbb{E}^{\mathbb{P}}[\int_0^T U^1(t, c_t^L) dt | \sigma(L)], \tag{4.3.3}$$

on all $(\mathcal{G}_t^L)$-progressively measurable strategies $c^L(.)$ under the budget constraint:

$$\mathbb{E}^{\mathbb{Q}^L}[\int_0^T e^{-\int_0^t r_s ds} c_t^L dt | \sigma(L)] = \mathbb{E}^{\mathbb{P}}[\int_0^T H_t^L c_t^L dt | \sigma(L)] \le x_0^L,$$

where x_0^L is a $\sigma(L)$-measurable random variable. This budget constraint comes from the fact that $\frac{X_t^L}{S_t^0} - \int_0^t \frac{c_s^L}{S_s^0} ds$ is a positive supermartingale.
In order to express the solution of this maximization problem, we define the function $y \to \mathcal{X}^L(y)$, which depends on the realization of L such that:

$$\mathcal{X}^L(y) = \mathbb{E}\left[\int_0^T I_1(t, y H_t^L) H_t^L | \sigma(L)\right].$$

We also denote by:

$$x \to \mathcal{Y}^L(x)$$

the inverse function of the function $\mathcal{X}^L(.)$. For a fixed y, $\mathcal{Y}^L(y)$ is a $\sigma(L)$-measurable random variable. It is proven in Hillairet [Hil05] that the Lagrange multiplier $\mathcal{Y}^L(x_0^L)$ is $\sigma(L)$-measurable (by monotony). Then the solution of the maximization problem is, for all $0 \le t \le T$ the $\mathbb{G}$-measurable optimal consumption path:

$$c_t^{L,*} = I_1(t, \mathcal{Y}^L(x_0^L) H_t^L). \tag{4.3.4}$$

Taking this last equation at time 0 gives the relation between the initial consumption and the Lagrange mutiplier $\mathcal{Y}^L(x_0^L)$:

$$\mathcal{Y}^L(x_0^L) = U_c^1(0, c_0^{L,*}). \tag{4.3.5}$$

We recall that $V^L(x_0^L)$ is the value function of the maximization problem (4.3.3), which is also $\sigma(L)$-measurable:

$$V^L(x_0^L) = \mathbb{E}\left[\int_0^T U^1(t, I_1(t, \mathcal{Y}^L(x_0)H_t^L))dt|\sigma(L)\right].$$

A consequence of this is that:

$$\mathbb{E}[V^L(x_0^L)] = \mathbb{E}\left[\int_0^T U^1(t, I_1(t, \mathcal{Y}^L(x_0)H_t^L))dt\right].$$

Proposition 4.3.1

- *If L is independent from $\mathbb{F}$, then $H_t^L = H_t^0$ for all $0 \le t \le T$. The impact of the parameter L has an impact on the consumption process c_t^L only through the initial consumption c_0^L. More precisely, the optimal consumption path is given by:*

$$c_t^{L,*} = I_1(0, H_t^0 U_c^1(0, c_0^{L,*})),$$

- *If L is not independent from $\mathbb{F}$, then $H_t^L \neq H_t^0$ and the impact of the parameter L on the consumption process c_t^L comes not only from c_0^L but also from the change of probability Z_t^L.*

Proof. If L is independent from $\mathbb{F}$, then $H_t^L = H_t^0$ for all $0 \le t \le T$. Because H_t^0 is independent from $\sigma(L)$, we have, for all $y \in \mathbb{R}$:

$$\mathcal{X}^L(y) = \mathbb{E}[\int_0^T I_1(t, yH_t^0)H_t^0|\sigma(L)] = \mathbb{E}[\int_0^T I_1(t, yH_t^0)H_t^0] = \mathcal{X}(y).$$

Thus $\mathcal{Y}^L \equiv \mathcal{Y}$, and $\mathcal{Y}^L$ does not depend on L. Thus, replacing in equations (4.3.4) and (4.3.5), the optimal consumption path is:

$$c_t^{L,*} = I_1(0, H_t^0 U_c^1(0, c_0^{L,*})),$$

for all $0 \le t \le T$. Thus, the optimal consumption path depends on L only through $c_0^{L,*}$. ■

Remark 4.3.2 *If L is independent from $\mathbb{F}$, then $H^L(.) \equiv H^0(.)$, thus:*

$$\begin{aligned}
V^L(x_0^L) &= \mathbb{E}\left[\int_0^T U^1(t, I_1(t, \mathcal{Y}(x_0^L)H_t^0))dt|\sigma(L)\right] \\
&= \mathbb{E}\left[\int_0^T U^1(t, I_1(t, \mathcal{Y}(x)H_t^0))dt|\sigma(L)\right]\Big|_{x=x_0^L} \\
&= \mathbb{E}\left[V(x)\right]\big|_{x=x_0^L}.
\end{aligned}$$

Thus $\mathbb{E}[V^L(x_0^L)] = \int_{\mathbb{R}} V(x_0^l)d\mu^L(l)$. If L is independant from $\mathbb{F}$, the same supconvolution relation as in holds. For $y > 0$, $\mathbb{E}[\tilde{V}^L(y)] = \tilde{V}(y)$.

In order to conclude this section, we examine the impact of L on the yield curve.

Similarly to Proposition 3.5.1 of Chapter 3, we characterize interest rates dynamics by "zero-coupon" prices defined by:

$$B^L(t,T) := \mathbb{E}^{\mathbb{Q}^L}\left[\exp(-\int_t^T r_s ds)|\mathcal{F}_t\right], \text{ for } 0 \leq t \leq T \leq T^H.$$

This quantity must be compared to zero-coupons in a complete market (the yield curve of an agent acting in a complete market with no information on L). We have the following result:

Proposition 4.3.2 *It is possible to express $B^L(t,T)$ as a function of $B(t,T)$ (zero-coupon price in a complete market). This measures the impact of the random variable on zero-coupon bond dynamics. For all $0 \leq t \leq T \leq T^H$ (where T^H is a time horizon):*

$$B^L(t,T) = B^{GOP}(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{E}^{-1}\left(-\int_t^T \langle \rho_s^L(L), dW_s\rangle\right)|\mathcal{F}_t\right],$$

where $\mathbb{Q}_T$ is the forward neutral probability measure, defined by:

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}}|_{\mathcal{F}_T} = \frac{e^{-\int_0^T r_s ds}}{B(0,T)}. \tag{4.3.6}$$

Proof. Using the expression of $B(t,T)$, the zero-coupon price in a complete market, we find:

$$\begin{aligned}
B^L(t,T) &= \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s ds}\mathcal{E}\left(-\int_t^T \langle \theta_s + \rho_s^L(L), d\widetilde{B}_s^L\rangle\right)|\mathcal{F}_t\right] \\
&= B(t,T)\frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{E}\left(-\int_t^T \langle \rho_s^L(L), d\widetilde{B}_s^L\rangle\right)|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right]} \\
&= B(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{E}\left(-\int_t^T \langle \rho_s^L(L), d\widetilde{B}_s^L\rangle\right)|\mathcal{F}_t\right],
\end{aligned}$$

where $\mathbb{Q}_T$ is the forward neutral probability measure, with density relatively to $\mathbb{Q}$ given by:

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}}|_{\mathcal{F}_T} = \frac{e^{-\int_0^T r_s ds}}{B(0,T)}.$$

This expression can be simplified, using the definition of the $\mathbb{Q}^L$-Brownian motion $\widetilde{B}_t^L = W_t - \int_0^t \rho_s^L(L)ds$, and becomes, in terms of the $\mathbb{P}$-Brownian motion $(W_t)_{t\geq 0}$:

$$\begin{aligned}
B^L(t,T) &= B(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{E}^{-1}\left(\int_t^T \langle \rho_s^L(L), dW_s\rangle\right)|\mathcal{F}_t\right] \\
&= B(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{E}\left(-\int_t^T \langle \rho_s^L(L), dW_s\rangle\right)\exp(\int_t^T (\rho_s^L(L))^2 ds)|\mathcal{F}_t\right].
\end{aligned}$$

Of course if the information drift $\rho^L \equiv 0$, the yield curve is not modified.

■

Because we have discussed the Ramsey Rule in the previous chapters, it is interesting to give the form of the Ramsey Rule in this framework.

Proposition 4.3.3

- *If L is independent from $\mathbb{F}$, then $H_t^L = H_t^0$. The Ramsey Rule is not changed.*
- *If L is not independent from $\mathbb{F}$, then $H_t^L \neq H_t^0$. We examine the impact on the Ramsey Rule:*

$$R^L(0,t) = \beta - \frac{1}{t}\log\mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_t^{L,*})}{u'(c_0^{L,*})}\right].$$

Proof. The solution of the maximization problem is:

$$\begin{aligned}
H_t^L &= \frac{U_c^1(t,c_t^{L,*})}{U_c^1(0,c_0^{L,*})} \\
&\exp(-\int_0^t r_s ds)\mathcal{E}(-\int_0^t \langle\theta_s, dW_s\rangle)p_t(L) = \frac{U_c^1(t,c_t^{L,*})}{U_c^1(0,c_0^{L,*})}.
\end{aligned}$$

Taking the expectation under the historical probability from both sides we get:

$$\mathbb{E}^{\mathbb{P}}\left[\exp(-\int_0^t r_s ds)\mathcal{E}(-\int_0^t \langle\theta_s, dW_s\rangle)p_t(L)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c^1(t,c_t^{L,*})}{U_c^1(0,c_0^{L,*})}\right].$$

Thus, using the definition of the new probability $\mathbb{Q}^L$ and of its density relatively to the historical probability $\mathbb{P}$, we get:

$$\mathbb{E}^{\mathbb{Q}^L}\left[\exp(-\int_0^t r_s ds)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c^1(t,c_t^{L,*})}{U_c^1(0,c_0^{L,*})}\right].$$

Then, we take the classical assumption that the utility from consumption function can be written as: $U^1(t,x) = \exp(-\beta t)u(x)$, for all t, x.
Then the yield curve $R^L(0,T)$ is such that, for all $0 \leq T \leq T^H$:

$$R^L(0,T) = \mathbb{E}^{\mathbb{Q}^L}\left[\exp(-\int_0^T r_s ds)\right].$$

And finally we obtain:

$$R^L(0,T) = \beta - \frac{1}{t}\log \mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^{L,*})}{u'(c_0^{L,*})}\right].$$

Of course, if L is independant from $\mathbb{F}$, then $H^L \equiv H^0$ and the Ramsey Rule is the same as in the complete market case.

■

It remains to generalize these questions for the investment/consumption problem in an incomplete market.

Chapter 5

Long-Term Interest Rates and dynamic utility functions

Contents

The purpose of this Chapter is to adapt the results obtained previously on long term interest rates (with standard utility functions) to the case of dynamic utility functions. The use of dynamic utility functions can be motivated by the following remark. An investor changes his behaviour and his preferences as time goes by. Because we consider a problem with a long horizon, it is also more likely that the investor will change his preferences during a longer time period. Moreover, we should recall that our long term study is motivated by the financing of ecological stakes, such as the global warming. Even if the magnitude of such climate changes is still uncertain, we can assume that the changes will be nevertheless important enough to modify the entire economy. It is clear that when it happens the investors will modify their preferences.
For our study it is thus crucial to take into account the fact that the agent can change his preferences during the observed time period, that is the agent's utility function changes along time. It is difficult to have a clear idea on how to specify the utility function. This motivates the use of dynamic utility functions.

Dynamic utility functions have been introduced by Musiela, Zariphopoulou, Rogers and studied by [Mra09, EM10].

In the following we give the definition of dynamic utility functions from terminal wealth, as they are presented for instance in [Mra09, EM10]. But they will not be our only concern. We will also introduce dynamic utility functions from consumption. Then we will define dual dynamic utility functions, as the Fenchel transforms of the previous utility functions.
Then we extend to the case of dynamic utility our method consisting in replacing the expected utility maximization problem from consumption and terminal wealth by an expected utility maximization problem from terminal wealth with a random horizon.
Using a result from [KM10b], we construct explicitly dynamic utility functions.
Throughout this chapter we will use results from [Mra09, EM10, KM10b].

First we recall the financial market that we consider throughout this chapter. It is the incomplete market described in Chapter 3. Consider the probability spce $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration of $\mathcal{F}$ satisfying the usual conditions, that is the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$. Consider $W(.)$ a N-dimensional standard Bronwnian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, N being the total number of sources of uncertainty in the market.

In this financial market there is one riskless asset with price $S^0(.)$ given by $dS^0_t = S^0_t r_t dt$ and M tradable risky assets (with $M < N$) in which we can invest. Their prices $S^i(.), i=1,\dots,M$ have the dynamics:

$$\frac{dS_t^i}{S_t^i} = \tilde{b}_t^i dt + \langle \tilde{\sigma}_t^i, dW_t \rangle,$$

where the drift process $\tilde{b}(.)$ is a $M \times 1$ column vector, and $\tilde{\sigma}(.)$ is a $M \times N$ volatility matrix, $\tilde{\sigma}^i(.)$ being its i-th line vector. We assume that $\sigma\sigma^{\mathbf{T}}(t,\omega)$ is an invertible adapted process.
We assume the existence of the risk premium process $\tilde{\theta}(.)$, a column vector of dimension N such that for all $0 \le t \le T$ it satisfies:

$$\tilde{\theta}_t = \tilde{\sigma}_t^{\mathbf{T}}(\tilde{\sigma}_t\tilde{\sigma}_t^{\mathbf{T}})^{-1}(\tilde{b}_t - r_t\mathbf{1}_M),$$

where we recall that $^{\mathbf{T}}$ denotes the transpose of a matrix. Integrability conditions on $\tilde{b}(.)$, $\tilde{\sigma}(.)$, $r(.)$ and $\tilde{\theta}(.)$ are assumed to be satisfied.

Consider self-financing portfolios. We call $\pi_t^i, i=1,\dots,M$ the fraction of wealth invested in each of the risky assets, and the volatility vector is $\kappa_t := \tilde{\sigma}_t^{\mathbf{T}}\pi_t$.
Then, using the self-financing equation, and taking the consumption process into account, the wealth $X^{x,c,\kappa}$ at time t of the agent is a solution of:

$$dX_t^{x,c\kappa} = -c_t dt + X_t^{x,c,\kappa} r_t dt + X_t^{x,c,\kappa}\langle \kappa_t, dW_t + \tilde{\theta}_t dt\rangle,\ X_0^{x,c,\kappa} = x.$$

In the following, we consider only positive wealth processes. We say that $(c,\kappa) \in \mathcal{A}(x)$ if the processes c and κ are $\mathbb{F}$-progressively measurable and for all $0 \le t \le T$, $X_t^{x,c,\kappa} \ge 0$ almost surely. In particular if $c \equiv 0$, we denote $X^{x,\kappa}(.)$ the associated wealth process.
Other assumptions and notations relative to the description of an incomplete market from Chapter 3 still hold. In particular, we consider the range of $\tilde{\sigma}_t^{\mathbf{T}}$. We call $\mathcal{K}_t$ the family of subvector of $\mathbb{R}^n$, such that: $\mathcal{K}_t = \sigma_t(\mathbb{R}^M)$. For all $0 \le t \le T$, by definition $\tilde{\theta}_t$ and $\kappa_t \in \mathcal{K}_t$, and we denote $\tilde{\theta}(.) \in \mathcal{K}$, $\kappa(.) \in \mathcal{K}$.

A process $H(.)$ is a state density process, if the process $H_t X_t^{x,c,\kappa} + \int_0^t H_s c_s ds$ is a $\mathbb{P}$-local martingale. Then there exists a progressively measurable process $\nu(.) \in \mathcal{K}^\perp$ (the orthogonal of $\mathcal{K}$ in $\mathbb{R}^N$) and the state price density process $H^\nu(.)$ is solution of the following stochastic differential equation, for all $0 \le t \le T$:

$$dH_t^\nu = -r_t dt - \langle \tilde{\theta}_t + \nu_t, dW_t\rangle, \text{ and } H_0^\nu = 1,\ \nu \in \mathcal{K}_t^\perp.$$

where $\tilde{\theta}(.)$ and $\nu(.)$ are orthogonal. As in Definition 3.1.1, we define the process $(Y_t^\nu(y))_{t\ge 0}$ such that for all $\nu \in \mathcal{K}^\perp$ and for all $y > 0$:

$$Y_t^\nu(y) = yH_t^\nu = y\exp(-\int_0^t r_s ds - \int_0^t \langle \tilde{\theta}_s + \nu_s, dW_s\rangle - \frac{1}{2}\int_0^t ||\tilde{\theta}_s + \nu_s||^2 du),$$

and $Y_0^\nu(y) = y$. This process appears naturally in the expression of the dual optimization problem. It is important to remark that $y \to Y_t^\nu(y)$ is a linear function of y. The process $(Y_t^\nu(1))_{t\geq 0}$ is called the minimal state price density process.
In this framework, for all $\nu \in \mathcal{K}^\perp$, the process $\int_0^t Y_s^\nu(y)c_s ds + Y_T^\nu(y)X_T^{x,c,\kappa}$ is a $\mathbb{P}$-local martingale. And for the optimal dual process, $X_t^{x,*}Y_t^*(y) + \int_0^t c_s^{\nu^*}Y_s^*(y)ds$ is a martingale.

The following notations are useful throughout this Chapter. For all admissible pairs of processes $(X_t^{x,c,\kappa}, c_t)_{t\geq 0}$, we say that $(X_t^{x,c,\kappa}, c_t)_{t\geq 0} \in \mathfrak{X}$. We use the same notation in the case of wealth only, where there is no consumption. Finally $\mathfrak{H}$ is the family of admissible state price density processes $(Y_t^\nu(y))_{t\geq 0}$.

5.1 Dynamic utility functions

Much work has been realized recently concerning dynamic utility function, that is utility function for which it is possible to represent in a dynamic way the preference changes of the representative agent as time goes by. These questions have been first motivated by the Merton problem (expected utility maximization from terminal wealth). For a detailed study of dynamic utility functions, see [Mra09], [EM10], or Zariphopoulou and Musiela.

We recall that a standard utility function is a concave increasing twice differentiable function $u : \mathbb{R}^+ \to \mathbb{R}^+$.

Definition 5.1.1 *A progressive utility function is a positive adapted random field $U(t,x)$ on $[0,+\infty[\times[0,+\infty[\times\Omega \to \mathbb{R}^+$ such that $U(t,.)$ is an increasing concave function.*

5.1.1 Dynamic utility functions from terminal wealth: state of the art

We give here the definition of a progressive dynamic utility function (from terminal wealth), such as it is given in [Mra09], page 135 and in [EM10] or also [KM10b, KM10a]. This is the definition which we use throughout this chapter. In this section, we consider an agent with no consumption, his wealth process is $X^{x,\kappa}(.)$.

Definition 5.1.2 *A dynamic utility function, is a positive adapted random field $U^2(t,x)$ such that $\mathbb{E}[U^2(t,x)] < +\infty$:*

- *Concavity assumption: for $t \geq 0$, $x > 0$: $x \to U^2(t,x)$, is a progressive utility function. At time 0, $U^2(0,x) = u^2(x)$.*
- *Consistency with the investment universe: for any admissible wealth process $X^{x,\kappa}(.) \in \mathfrak{X}$, and we have for all $0 \leq t \leq s \leq T$:*

$$\mathbb{E}[U^2(s, X_s^{x,\kappa})|\mathcal{F}_t] \leq U^2(t, X_t^{x,\kappa}).\ a.s.$$

- *Existence of optimal wealth: for any initial wealth x, there exists an optimal wealth process $X_t^{*,x}$ (with initial wealth $X_0^{*,x} = x$) such that for all $0 \leq t \leq s \leq T$: $\mathbb{E}[U^2(s, X_s^{*,x})|\mathcal{F}_t] = U^2(t, X_t^{*,x})$.*

Then $U^2(t,x)$ is a $\mathfrak{X}$-consistent dynamic utility function. For any admissible wealth process with initial wealth x, $t \to U^2(t, X_t^{x,\kappa})$ is a positive supermartingale, and a martingale for the optimal wealth $X^{*,x}(.)$.

5.1.2 Dynamic utility functions from consumption and terminal wealth

Throughout this work, we have underlined many times the importance of the consumption process. Similarly, in expected utility maximization problem, the consumption/investment problem, or even the utility maximization problem from consumption only were often used. Thus it is a natural step to extend the definition of dynamic utility functions to the case where the representative agent maximizes the expected utility of his consumption.
For these dynamic utilities from consumption and terminal wealth, here is the definition that we choose to consider:

Definition 5.1.3 *Dynamic utility from consumption and terminal wealth:*
Consider two progressive utility functions $U^1(t,.)$ and $U^2(t,.)$ such that $\mathbb{E}[U^2(t,x)] < +\infty$ and $\mathbb{E}[U^1(t,c)] < +\infty$. At time 0, $U^2(0,x) = u^2(x)$ and $U^1(0,c) = u^1(0,c)$ are standard utility functions.
Consistence with investment universe: for all admissible wealth processes and associated consumption processes $(X^{x,c,\kappa}(.), c(.)) \in \mathfrak{X}$. For all these test processes

$$U^2(t, X_t^{x,c,\kappa}) + \int_0^t U^1(s, c_s)ds, \tag{5.1.1}$$

is a supermartingale.
There exists an optimal pair $(X_t^{*,x}, c_t^*(x))_{t\geq 0} \in \mathfrak{X}$ for which it is is a martingale (in this expression $c_t^*(x)$ denotes the fact that the optimal consumption path depends on the initial wealth x). Then (U^1, U^2) is a pair of $\mathfrak{X}$-consistent dynamic utility functions.

Dynamic utility functions of investment and consumption have been studied by Berrier and Tehranchi [BT08]. However this definition is different from theirs, their utility functions are not required to be positive. Moreover, in our framework, wealth processes are not required to be discounted.

In Definition 5.1.3 we obtain a **dynamic preference structure** (U^1, U^2), characterizing the preferences of an agent in terms of dual dynamic utility functions. Or also, using the same terminology as [BT08], the functions (U^1, U^2) may be called a dynamic utility pair.
One can check that if $U^1 \equiv 0$, this definition satisfies the definition of dynamic utility functions from terminal wealth only (Definition 5.1.2).
However, the simplicity of the dual problem motivates the definition of dual dynamic utility functions.

5.1.3 Dual dynamic utility functions

To the dynamic utility functions from consumption and terminal wealth presented previously (they are concave) we associate their Fenchel transforms (which are convex). For any consistent dynamic utility function U^1 the dual consistent dynamic utility from consumption function is a positive random field $\tilde{U}^1$ on $[0, +\infty[\times[0, +\infty[\times\Omega \to \mathbb{R}^+$ is defined by:

$$\tilde{U}^1 : (t, y) = \sup_{c>0}\{U^1(t, c) - cy\}. \tag{5.1.2}$$

Similarly, to any consistent dynamic utility function from terminal wealth U^2, we associate the dual dynamic utility from terminal wealth function is a positive random field $\tilde{U}^2$ on $[0, +\infty[\times[0, +\infty[\times\Omega \to \mathbb{R}^+$ defined by:

$$\tilde{U}^2 : (t, z) = \sup_{x>0}\{U^2(t, x) - xz\}. \tag{5.1.3}$$

Throughout this chapter we call $\tilde{U}(t, x)$ a progressive dual utility function if it is a positive adapted random field on $[0, +\infty[\times[0, +\infty[\times\Omega \to \mathbb{R}^+$ such that for all $t \geq 0$, $\tilde{U}(t, .)$ is a convex decreasing function.

Proposition 5.1.1 *Let us call $\mathfrak{H}$ the family of state price density processes $(Y_t^\nu(y))_{t\geq 0}$. The random fields $(\tilde{U}^1, \tilde{U}^2)$ defined above are a pair of $\mathfrak{H}$-consistent dual dynamic utility functions. They satisfy the following properties:*

- *$\tilde{U}^1(t, .)$ and $\tilde{U}^2(t, .)$ are progressive dual utility functions.*

- *Consistence with the state price density. For all state price density processes $Y^\nu(y) \in \mathfrak{H}$, the following process is a submartingale:*

$$\tilde{U}^2(t, Y_t^\nu(y)) + \int_0^t \tilde{U}^1(s, Y_s^\nu(y))ds. \tag{5.1.4}$$

- *Existence of an optimum ν^*, such that $\tilde{U}^2(t, Y_t^*(y)) + \int_0^t \tilde{U}^1(s, Y_s^*(y))ds$ is a martingale.*

This is a ***dual dynamic preference structure*** $(\tilde{U}^1, \tilde{U}^2)$*. These functions are dual consistent dynamic utility function.*

The strength of this formulation is that the test processes are state price density processes. Even for an investment/consumption problem, in the dual case there is only one family of test processes (whereas wealth AND consumption were needed as test processes in Definition 5.1.3).

Proof. The dual utility functions $\tilde{U}^1$ and $\tilde{U}^2$ are defined as Fenchel transforms of U^1 and U^2, they are convex and decreasing. The next step of the proof is along the lines of the work of El Karoui and Mrad [EM10], and a proof of Berrier et al. [BRT07].
Let $X^*(.)$ and $c^*(.)$ be optimal processes. Thus, for any process $Y^\nu(y) \in \mathfrak{H}$, by definition of a Fenchel transform, for all $0 \leq t \leq T$:

$$\tilde{U}^2(t, Y_t^\nu(y)) \geq U^2(t, X_t^*) - Y_t^\nu(y) X_t^*,$$

and an analoguous inequality holds for $\tilde{U}^1$. Adding these two expressions, one gets:

$$\tilde{U}^2(t, Y_t^\nu(y)) + \int_0^t \tilde{U}^1(s, Y_s^\nu(y))ds \tag{5.1.5}$$

$$\geq U^2(t, X_t^{*,x}) + \int_0^t U^1(s, c_s^*)ds - (Y_t^\nu(y) X_t^{*,x} + \int_0^t Y_s^\nu(y) c_s^* ds). \tag{5.1.6}$$

In the remaining part of this proof for all $0 \leq u \leq t$ we denote the state price density process $Y_{u,t}^\nu(y) := y H_{u,t}^\nu$. For all u-attainable wealth η, we call $X_t^*(u, \eta)$ the optimal wealth process starting from η at time u. Using the martingale property of $U^2(t, X_t^*(u, \eta)) + \int_0^t U^1(s, c_s^*)ds$, and the inequality above (5.1.5), we have for all $0 \leq u \leq t \leq T$:

$$\begin{aligned}
&\mathbb{E}[\tilde{U}^2(t, Y_{u,t}^\nu(y)) + \int_u^t \tilde{U}^1(s, Y_{u,s}^\nu(y))ds|\mathcal{F}_u] \\
&\geq \mathbb{E}[U^2(t, X_t^*(u, \eta)) + \int_u^t U^1(s, c_s^*)ds - (Y_t^\nu(y) X_t^*(u, \eta) + \int_u^t Y_s^\nu(y) c_s^* ds)|\mathcal{F}_u] \\
&\geq U^2(u, \eta) - y\eta.
\end{aligned}$$

Because our portfolio constraints lie in subvector spaces, if η is a u-attainable wealth, then for all $\lambda > 0$, $\eta\lambda$ is also an u attainable wealth, only with a different initial condition. The following inequality holds for all $\lambda > 0$:

$$\mathbb{E}[\tilde{U}^2(t, Y^\nu_{u,t}(y)) + \int_u^t \tilde{U}^1(s, Y^\nu_{u,s}(y))ds|\mathcal{F}_u] \geq U^2(u, \eta\lambda) - y\eta\lambda.$$

It is interesting to choose a particular λ such that $\lambda\eta = (U^2_x)^{-1}(u, y)$. Then the inequality above becomes:

$$\begin{aligned}&\mathbb{E}[\tilde{U}^2(t, Y^\nu_{u,t}(y)) + \int_u^t \tilde{U}^1(s, Y^\nu_{u,s}(y))ds|\mathcal{F}_u]\\ &\geq U^2(u, (U^2_x)^{-1}(u, y)) - y(U^2_x)^{-1}(u, y) = \tilde{U}^2(u, y).\end{aligned}$$

where the last equality comes from the definition of a Fenchel transform. This proves Point 2 of the Proposition.
Now, let (U^1, U^2) be a pair of dynamic utility functions. Then, let y be u-admissible. Then there exists an u-admissible wealth η such that $U^2_x(u, \eta) = y$. Let $(X^*_t(u, \eta), c^*_t)_{t\geq u}$ the optimal wealth and consumption processes with $X^*_u = \eta$. Now denote by $(Y^*_{u,t}(y))_{t\geq u}$ the process defined by:

$$Y^*_{u,t}(y) := U^2_x(t, X^*_t(u, \eta)) = U^2_x(t, X^*_t(u, (U^2_x)^{-1}(u, y))). \tag{5.1.7}$$

Then, the same kind of proof as Berrier et al. in Theorem 3.1 of [BT08] gives the relation between the optimal consumption path and $(Y^*_{u,t}(y))_{t\geq u}$, that is:

$$U^1_c(t, c^*_t) = Y^*_{u,t}(y). \tag{5.1.8}$$

Then, using equation (5.1.7), one gets $(U^2_x)^{-1}(t, Y^*_{u,t}(y)) = X^*_t(u, (U^2_x)^{-1}(u, y))$, thus:

$$\tilde{U}^2(t, Y^*_{u,t}(y)) = U^2(t, X^*_t(u, (U^2_x)^{-1}(u, y))) - Y^*_{u,t}(y)X^*_t(u, (U^2_x)^{-1}(u, y)). \tag{5.1.9}$$

On the other hand, a similar kind of calculation for the utility from consumption gives, using equation (5.1.8): $(U^1_c)^{-1}(t, Y^*_{u,t}(y)) = c^*_t$, thus for all $t \geq u$ $\tilde{U}^1(t, Y^*_{u,t}(y)) = U^1(t, c^*_t) - Y^*_{u,t}(y)c^*_t$. Thus:

$$\int_u^t \tilde{U}^1(s, Y^*_{u,s}(y))ds = \int_u^t U^1(s, c^*_s)ds - \int_u^t Y^*_{u,s}(y)c^*_s ds. \tag{5.1.10}$$

Putting equations (5.1.9) and (5.1.10) together gives:

$$\begin{aligned}\tilde{U}^2(t, Y^*_{u,t}(y)) + \int_u^t \tilde{U}^1(s, Y^*_{u,s}(y))ds &= U^2(t, X^*_t(u, (U^2_x)^{-1}(u, y))) + \int_u^t U^1(s, c^*_s)ds\\ &- Y^*_{u,t}(y)X^*_t(u, (U^2_x)^{-1}(u, y)) - \int_u^t Y^*_{u,s}(y)c^*_s ds.\end{aligned}$$

With similar arguments as in [EM10], using the fact that (U^1, U^2) is a $\mathfrak{X}$-consistent dynamic utility pair, using the martingale property of $(U^2(t, X_t^*(u,\eta)) + \int_u^t U^1(s, c_s^*)ds)_{t\geq u}$ and of $(U_x^2(t, X_t^*(u,\eta))X_t^*(u,\eta) + \int_u^t U_c^1(s, c_s^*)c_s^* ds)_{t\geq u}$ and by definition of $(Y_{u,t}^*(y))_{t\geq u}$, we get that $(\tilde{U}^2(t, Y_{u,t}^*(y)) + \int_u^t \tilde{U}^1(s, Y_{u,s}^*(y))ds)_{t\geq u}$ is a martingale. This concludes the proof. ■

Remark 5.1.1 *Let us consider U^2, which is a consistent dynamic utility from terminal wealth only satisfying Definition 5.1.2. Its dual function satisfies the following definition which can be found in the work of El Karoui and Mrad [Mra09, KM10b, EM10]. A dual consistent dynamic utility function satisfies the following properties: $\tilde{U}(t,y)$ is a convex decreasing function.*
Consistence with the investment universe. Let us call $\mathfrak{H}$, the family of state price density processes. For all state price density processes $Y_t^\nu(y) \in \mathfrak{H}$, $\tilde{U}(t, Y_t^\nu(y))$ is a submartingale.
Existence of an optimum ν^*: the process $\tilde{U}(t, Y_t^*(y))$ is a martingale.
But it is a dual consistent dynamic utility function satisfying the conditions given in Definition 5.1.1: both definitions are consistent if $U^1 \equiv 0$.

Optimal processes: Let us recall a part of a Theorem from [KM10a] (Theorem 3.3). That is, for any $Y^\nu(y) \in \mathfrak{H}$, $\tilde{U}(t, Y_t^\nu(y))$ is a submartingale and there exists a dual optimal choice $\nu_t^*(y)$. For any $y > 0$, the process $Y_t^*(y) := U_x(t, X_t^*(\tilde{U}_y(y)))$ is an optimal dual process and satisfies the SDE:

$$\frac{dY_t^*(y)}{Y_t^*(y)} = r_t dt - \langle \nu_t^*(Y_t^*(y)) + \tilde{\theta}_t, dW_t \rangle, \ Y_0^*(y) = y, \tag{5.1.11}$$

where $\nu_t^*(Y_t^*(y))$ denotes the fact that ν^* depends on $Y_t^*(y)$. We do not give the expression of $\nu_t^*(Y_t^*(y))$ here, we only underline the fact that it depends on t and y. Here ν^* depends on the initial condition y but not on the maturity T, in contrast to the end of Chapter 3.

Consider an optimal state price density process $Y_t^*(y)$ with initial value y. Consider an agent starting from an initial wealth x. His optimal wealth is $X_t^*(x)$ and his optimal consumption is denoted here $c_t^*(c_0^*)$, to underline the link with initial value. In Theorem above, we have the following relations:

$$U_x^2(t, X_t^*(x)) = Y_t^*(y). \tag{5.1.12}$$

$$U_c^1(t, c_t^*(c_0^*)) = Y_t^*(y). \tag{5.1.13}$$

This shows a strong link between optimal consumption, optimal wealth and optimal state price density process.

Reconstruction of a dynamic utility function Given a state price density process $Y_t^*(y)$, given a standard preference structure (a pair of standard utility functions) $(u^1(t,c), u^2(x))$, and given a pair of optimal processes $(X_t^*(x), c_t^*(c_0^*))$, and assuming that these two processes are monotonous relatively to their initial value, one can construct a pair of dynamic utility functions.
The processes given in equations (5.1.12,5.1.13) are natural candidates for the reconstruction of a utility function between 0 and t. With the assumption that the processes $X_t^*(x)$ and $c_t^*(c_0^*)$ are monotonous relatively to their initial value, it is possible to define the inverse processes $z \to (X_t^*)^{-1}(z)$ and $z \to (c_t^*)^{-1}(z)$.
In [Mra09] and related papers, a construction of $U_x^2(t,x)$, the derivative relatively to x of a dynamic utility function from wealth is given by:

$$U_x^2(t,x) = Y_t^*(u_x^2((X_t^*)^{-1}(x))). \tag{5.1.14}$$

Integrating this expression gives a construction of $U^2(t,x)$, a dynamic utility function from wealth.
Along the same lines, we propose the following construction of the derivative of the dynamic utility function from consumption, for all $t \geq 0$:

$$U_c^1(t,c) = Y_t^*(u_c^1(t,(c_t^*)^{-1}(c_0^*))). \tag{5.1.15}$$

We treat the case of the construction of a dynamic utility function in the $\mathbb{G}$-market in the section 5.3.

5.2 Utility maximization with dynamic utility functions in the $\mathbb{G}$-market

5.2.1 Dynamic utility functions in the $\mathbb{G}$-market

Along the same lines as in Chapter 4, we introduce the filtration $\mathcal{G}$. We use the same notations as in Chapter 4. However here the utility functions are dynamic instead of standard utility functions. More precisely let us recall that $0 < \zeta < \infty$ is a positive random variable. Then let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the minimal filtration which makes τ a $\mathbb{D}$-stopping time, i.e. $\mathcal{D}_t = \mathcal{D}_{t+}^0$ with $\mathcal{D}_t^0 = \sigma(\zeta \wedge t)$. Then we consider the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ such that:

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t,$$

and assuming that $\mathcal{G}_0$ contains all null sets of $\mathcal{G}_\infty$.
Moreover, any $\mathcal{G}_t$ measurable random variable Ψ_t^G may be represented as:

$$\Psi_t^G = \Psi_t^F \mathbf{1}_{t<\zeta} + \Psi_t(\zeta)\mathbf{1}_{\zeta\leq t},$$

where Ψ_t^F is an $\mathcal{F}_t$ measurable random variable and $\Psi_t(\zeta)$ is $\mathcal{F}_t \otimes \sigma(\zeta)$ measurable.
We define the process Φ_t in terms of the conditional distribution of ζ given $\mathcal{F}_\infty$:

$$\mathbb{P}[\zeta > t | \mathcal{F}_\infty] = e^{-\Phi_t}.$$

The process Φ_t is assumed to be continuous, increasing, $\mathbb{F}$ adapted and such that $\Phi_\infty = \infty$. This kind of framework apprears usually in credit modelling, see for instance the work on what happens after default of [EJJ10].
Moreover, we assume in this Chapter, as in Chapter 4, that Φ_t is differentiable relatively to t almost surely and we denote, for all $0 \leq t \leq T$:

$$\Phi_{t,T} = \Phi_T - \Phi_t = \int_t^T \varphi_s ds.$$

We assume that the intensity φ_t is strictly positive almost surely for all $t > 0$. Indeed, in the following paragraphs the term $\frac{1}{\varphi_t}$ appears and it has to be well defined.
We recall the following definitions from Chapter 4. Consider a representive agent starting from an initial consumption x and taking a portfolio $\kappa(.)$ and consumption $c(.)$. We define the $\mathbb{F}$-progressively measurable process $\check{X}^{x,c,\kappa}(.)$, for all $0 \leq t \leq T$ as:

$$X_t^{x,c,\kappa} = \check{X}_t^{x,c,\kappa} e^{\Phi_t}.$$

Thus for $(c, \kappa) \in \mathcal{A}(x)$, by definition $\check{X}^{x,c,\kappa}(.)$ is positive $\mathbb{P}$-almost surely. And for all $u \geq 0$, we define the $\mathbb{F}$ progresively measurable process $\check{c}(.)$ as:

$$c_u = \check{c}_u e^{-\Phi_u} \varphi_u.$$

At each date t, a certain quantity of wealth is put in reserve (in case of the event ζ). The integrability of $\check{c}_\zeta \mathbf{1}_{\zeta \leq t}$ is discussed in Chapter 4. We recall Definition 4.1.1, which gives the expression of the wealth process in the $\mathbb{G}$-market.
We consider a representative agent with initial wealth x, portfolio $\kappa(.)$ and consumption $c(.)$ and $X^{x,c,\kappa}(.)$ the associated wealth process. Then we define the process $X^{\mathbb{G},x,\kappa}(.)$ as:

$$\begin{aligned} X_t^{\mathbb{G},x,\kappa} &= \check{X}_t^{x,c,\kappa} \text{ for } t < \zeta \\ X_\zeta^{\mathbb{G},x,\kappa} &= \check{c}_\zeta. \end{aligned}$$

The process $X^{\mathbb{G},x,\kappa}(.)$ defined here has one jump at ζ.
With this formulation, we interpret the consumption rate as a certain quantity of cash. As it was explained in Chapter 4, this quantity of cash is put

aside in case of an unpredictable bad event (also called default, if we choose to use the credit risk vocabulary). For $0 \leq t < \zeta$, the portfolio is managed in the classical financial way (consumption/investment). As we have mentioned in Section 4.1.4, the wealth process is self-financing until date ζ^- only. Then at random date ζ, the investor no longer invests in the various assets. He only uses his supplies in cash.
It is important to notice that because the consumption rate is by definition positive almost surely, and so is $\check{X}^{x,c,\kappa}(.)$, for all $\mathbb{F}$ progressively measurable admissble strategies $(c,\kappa) \in \mathcal{A}(x)$, for all $t \geq 0$, we have:

$$X_t^{\mathbb{G},x,\kappa} \geq 0, \mathbb{P} - a.s.$$

Given a dynamic preference structure (U^1, U^2), we define the functions $\check{U}^1(t,c)$ and $\check{U}^2(t,x)$, for all $0 \leq t \leq T$ as:

$$\check{U}^1(t,c) = U^1(t, ce^{-\Phi_t})e^{\Phi_t}(\varphi_t)^{-1},\, \check{U}^2(t,x) = U^2(t, xe^{-\Phi_t})e^{\Phi_t}.$$

It is possible to define a function $U^{\mathbb{G}}(t,x)$ in the $\mathbb{G}$-market, as in Chapter 4. Here the only difference is that the building blocks (U^1, U^2) are already stochastic.

Theorem 5.2.1 *Let us consider a representative agent with a dynamic preference stucture (U^1, U^2). Let $0 < \zeta < \infty$ be a random variable. Using the previous definition of $\check{U}^1$ and $\check{U}^2$, we define $U^{\mathbb{G}}(t,x)$ such that for all $x > 0$:*

$$U^{\mathbb{G}}(t,x) = \check{U}^2(t,x)\mathbf{1}_{t<\zeta} + \check{U}^1(\zeta,x)\mathbf{1}_{\zeta \leq t} \ a.s.$$

For all admissible wealth processes $X^{\mathbb{G},x,\kappa}(.)$ we say that $X^{\mathbb{G},x,\kappa}(.) \in \mathfrak{X}^{\mathbb{G}}$. Then $U^{\mathbb{G}}(t,x)$ is a $\mathfrak{X}^{\mathbb{G}}$ ***consistent dynamic utility function in the $\mathbb{G}$-market****. That is, $U^{\mathbb{G}}$ is a positive random field, such that $U^{\mathbb{G}}(t,.)$ is increasing an concave, and $\mathbb{E}[U^{\mathbb{G}}(t,x)] < +\infty$ and at time 0, $U^{\mathbb{G}}(t,x) := u^2(x)$, a standard utility function. And for all test processes $X^{\mathbb{G},x,\kappa}(.) \in \mathfrak{X}^{\mathbb{G}}$, the process*

$$t \to U^{\mathbb{G}}(t, X_t^{\mathbb{G},x,\kappa})\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta \leq t}, \tag{5.2.1}$$

is a supermartingale and there exists an optimum for which it is a martingale.

Proof. One can check that for a fixed $t > 0$, $U^{\mathbb{G}}(t,.)$ is an increasing concave function. Integrability of $U^{\mathbb{G}}(t,.)$ comes from the integrability of U^1, U^2 and the definition of ζ. At time 0, $U^{\mathbb{G}}(0,x) = u^2(x)$. But just after time 0, we have $U^{\mathbb{G}}(t,x) = U^2(t,x)\mathbf{1}_{t<\zeta} + U^1(\zeta,x)\mathbf{1}_{\zeta \leq t}$. Thus, the initial condition of the utility from consumption function has to be given: $u^1(\theta,x)$, which will be taken at values $u^1(\theta,x)|_{\theta=\zeta}$.

For all test processes in the $\mathbb{G}$-market, such that $X^{\mathbb{G},x,\kappa}(.) \in \mathfrak{X}^{\mathbb{G}}$, for all $0 \leq t \leq s$, along the same lines as the proofs given in Chapter 4 (Remark 4.1.1):

$$\begin{aligned}
&\mathbb{E}[U^{\mathbb{G}}(s, X_s^{\mathbb{G}})\mathbf{1}_{s<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta \leq s}|\mathcal{G}_t] \\
&= \mathbb{E}[U^2(s, X_s^{x,c,\kappa}) + \int_t^s U^1(u, c_u)du|\mathcal{F}_t]e^{\Phi_t}\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta \leq t} \\
&\leq U^2(t, X_t^{x,c,\kappa})e^{\Phi_t}\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G}})\mathbf{1}_{\zeta \leq t} \text{ a.s.,}
\end{aligned}$$

where the last inequality comes from the definition of a dual dynamic pair (U^1, U^2). And finally we use the fact that:

$$U^2(t, X^{x,c,\kappa})e^{\Phi_t}\mathbf{1}_{t<\zeta} = \check{U}^2(t, \check{X}^{x,c,\kappa})\mathbf{1}_{t<\zeta} = U^{\mathbb{G}}(t, X_t^{\mathbb{G},x,\kappa})\mathbf{1}_{t<\zeta}.$$

This shows that $[U^{\mathbb{G}}(t, X_t^{\mathbb{G},x,\kappa})\mathbf{1}_{t<\zeta} + U^{\mathbb{G}}(\zeta, X_\zeta^{\mathbb{G},x,\kappa})\mathbf{1}_{\zeta \leq t}$ is a supermartingale. For the same reasons it is a martingale at the optimum. Thus we call $U^{\mathbb{G}}(t, .)$ a $\mathfrak{X}^{\mathbb{G}}$**consistent dynamic utility function in the $\mathbb{G}$-market**.

■

However, the dual approach seems more simple, and this is the one we priviledge in the following section.

5.2.2 The dual problem with dynamic utility functions in the $\mathbb{G}$ market

Our method consists in writing the dual expected utility maximization problem from consumption and terminal wealth as a dual expected utiliy maximization problem from terminal wealth only. In Chapter 4, we followed this method with standard utility functions. Here we follow the same lines with dynamic utility functions. We define the functions $\check{\tilde{U}}_1(t, y)$ and $\check{\tilde{U}}_2(t, y)$, for all $0 \leq t \leq T$ as:

$$\check{\tilde{U}}^1(t, y) = \tilde{U}^1(t, y)e^{\Phi_t}(\varphi_t)^{-1}, \; \check{\tilde{U}}^2(t, y) = \tilde{U}^2(t, y)e^{\Phi_t}.$$

Similarly to Chapter 4 (definition 4.1.4), it is possible to define a dynamic dual utility function in the $\mathbb{G}$-market.

Definition 5.2.1 *Let consider a representative agent with a preference stucture (U^1, U^2). Let $\zeta > 0$ be a random variable independant from $\mathcal{F}_\infty$. Using the previous definition of $\check{\tilde{U}}^1$ and $\check{\tilde{U}}^2$, we define $\tilde{U}^{\mathbb{G}}(t, y)$ such that for all $y > 0$, for all $0 \leq t \leq T$:*

$$\tilde{U}^{\mathbb{G}}(t, y) = \check{\tilde{U}}^2(t, y)\mathbf{1}_{t<\zeta} + \check{\tilde{U}}^1(\zeta, y)\mathbf{1}_{\zeta \leq t}.$$

On can check that with this definition, we have:

$$\tilde{U}^{\mathbb{G}}(t,y) = \inf_{x>0}\{U^{\mathbb{G}}(t,x) - xy\}$$

By definition, the functions $\check{\tilde{U}}^1$ and $\check{\tilde{U}}^2$ satisfy the same conditions as $\tilde{U}^1$ and $\tilde{U}^2$, that is they are dual consistent dynamic utility functions.

Proposition 5.2.1 *The dynamic functions $\tilde{U}^{\mathbb{G}}(t,.)$ are convex and decreasing. For all test processes $(Y_t^{\nu}(y))_{t\geq 0} \in \mathfrak{H}$, the process $t \to \tilde{U}^{\mathbb{G}}(t\wedge\zeta, Y^{\nu}_{t\wedge\zeta}(y))$ is a submartingale, that is:*

$$\mathbb{E}[\tilde{U}^{\mathbb{G}}(T\wedge\zeta, Y^{\nu}_{T\wedge\zeta}(y))|\mathcal{G}_t] \geq \tilde{U}^{\mathbb{G}}(t\wedge\zeta, Y^{\nu}_{t\wedge\zeta}(y)).$$

And for the optimal dual process ν^*, the process $t \to \tilde{U}^{\mathbb{G}}(t\wedge\zeta, Y^*_{t\wedge\zeta}(y))$ is a $\mathbb{G}$-martingale, that is, for all $t \leq T$:

$$\mathbb{E}[\tilde{U}^{\mathbb{G}}(T\wedge\zeta, Y^{*}_{T\wedge\zeta}(y))|\mathcal{G}_t] = \tilde{U}^{\mathbb{G}}(t\wedge\zeta, Y^{*}_{t\wedge\zeta}(y)).$$

Again, the dual case is the most simple. The test processes $Y^{\nu}(y)$ are the same as in the $\mathbb{F}$-market. In the previous section, it was necessary to define a wealth process $X^{\mathbb{G}}(.)$ in the $\mathbb{G}$-market with a jump, as a test process for dynamic utility functions $U^{\mathbb{G}}$. But here in the dual case, state prices density processes $Y^{\nu}(y)$ are test processes.
Moreover, it is much more simple to introduce the $\mathbb{G}$-market and show martingale properties on $U^{\mathbb{G}}$, which depends only on $X^{\mathbb{G}}$ than it was for the dynamic pair (U^1, U^2), which involved both wealth and consumption processes.

Proof. As the Fenchel tranform of the the dynamic utility function $U^{\mathbb{G}}$, $\tilde{U}^{\mathbb{G}}$ is convex and decreasing. Then, along the same lines as in Proposition 5.1.1, it can be proven that for all test processes in $\mathfrak{H}$:

$$\tilde{U}^{\mathbb{G}}(t, Y_t^{\nu}(y))\mathbf{1}_{t<\zeta} + \tilde{U}^{\mathbb{G}}(\zeta, Y_\zeta^{\nu}(y))\mathbf{1}_{\zeta\leq t}$$

is a $(\mathbb{G},\mathbb{P})$ submartingale and a martingale for the optimal state price density process. The process above can be rewritten as $t \to \tilde{U}^{\mathbb{G}}(t\wedge\zeta, Y^{\nu}_{t\wedge\zeta}(y))$ and the properties follow. ■

5.3 Construction of a dynamic utility function

In this section, we construct a dynamic utility function in the $\mathbb{G}$-market. For more details concerning the construction of dynamic utility functions, we

refer to the work of [Mra09].
Because the use of the $\mathbb{G}$-market for utility maximization is one of our new results, we provide here a way to construct dynamic utility functions in the $\mathbb{G}$-market.

Theorem 5.3.1 *Given an optimal state price density process $(Y_t^*(y))_{t\geq 0}$, a pair of optimal wealth/consumption processes $(X_t^{*,x}, c_t^*(x))_{t\geq 0} \in \mathfrak{X}$, a standard preference structure $(u^1(t,c), u^2(x))$, the derivative of $U_x^{\mathbb{G}}$ relatively to x of a dynamic utility function $U^{\mathbb{G}}$ in the $\mathbb{G}$-market is given by:*

$$U_x^{\mathbb{G}}(t,x) = Y_t^*(u_x^2((X_t^{*,x})^{-1}(x)))\mathbf{1}_{t<\zeta} + Y_\zeta^*(u_c^1(\zeta, (c_\zeta^*)^{-1}(x)))\mathbf{1}_{\zeta\leq t}.$$

Proof. The optimal state price density process $(Y_t^*(y))_{t\geq 0}$ does not jump, it is the same form of state price density as in an $\mathbb{F}$-market. We call u_c^1 the derivative of the standard utility from consumption function and u_x^2 the derivative of the standard utility from terminal wealth function. Using equations (5.1.7) and (5.1.8), we have: $Y_{u,t}^*(y) = U_x^2(t, X_t^*(u, (U_x^2)^{-1}(u,y)))$ and $U_c^1(t, c_t^*) = Y_{u,t}^*(y)$. Then, similarly to section 5.1.3, we construct the derivative of $U^{\mathbb{G}}$ relatively to x, for all $t \geq 0, x \geq 0$:

$$U_x^{\mathbb{G}}(t,x) = Y_t^*(u_x^2((X_t^{*,x})^{-1}(x)))\mathbf{1}_{t<\zeta} + Y_\zeta^*(u_c^1(\zeta, (c_\zeta^*)^{-1}(x)))\mathbf{1}_{\zeta\leq t}.$$

Integrating this last expression relatively to x, an example of function $U^{\mathbb{G}}$ is obtained. ∎

5.4 Dynamic utility functions and yield curve

5.4.1 Expression of the yield curve

The difference induced by the use of dynamic utility functions appears more clearly in this section. Here similarly to Section 3.5 of Chapter 3, we express the yield curve an zero-coupon bond dynamics, and see the differences induced by dynamic utility functions.
We recall that the state price density process $Y_t^*(y)$ is given by:

$$\frac{dY_t^*(y)}{Y_t^*(y)} = r_t dt - \langle \nu_t^*(Y_t^*(y)) + \tilde{\theta}_t, dW_t\rangle,\ Y_0^*(y) = y,$$

Thus, the optimal dual process ν^* depends on y. In this section, we denote it $\nu^*(y)$. In this framework, the yield curve depends also on $\nu^*(y)$, this is why we denote it $R^{\nu^*(y)}$.

Theorem 5.4.1 *The yield curve is given by:*

$$R_T^{\nu^*(y)}(s) = -\frac{1}{s}\log\mathbb{E}^{\mathbb{P}}[\frac{1}{y}Y^*_{T,T+s}(y)|\mathcal{F}_T] = -\frac{1}{s}\log\mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[exp(-\int_T^{T+s} r_s ds)|\mathcal{F}_T], \tag{5.4.1}$$

where we recall that $\nu^*(y) \in \mathcal{K}^\perp$ and $\tilde{\theta}(.) \in \mathcal{K}$ is the minimal risk premium process. The probability $\mathbb{Q}^{\nu^*(y)}$ is defined as:

$$\frac{d\mathbb{Q}^{\nu^*(y)}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}\big(-\int_0^t \langle\tilde{\theta}_s + \nu^*_s(Y^*_s(y)), dW_s\rangle\big),$$

Thus, the important consequence is that the yield curve depends on y the wealth in the economy.

Moreover, we have seen earlier in this Chapter the relation linking dynamic utility from consumption, the optimal consumption process and the state price density process $Y^*_t(y)$ with initial value y:

$$U^1_c(t, c^*_t) = Y^*_t(y),$$

and at time 0: $U^1_c(0, c^*_0) = y$. Then, rewriting (5.4.1) gives a Ramsey Rule, linking the yield curve with **marginal dynamic utility from consumption** and the optimal consumption path:

$$R_0^{\nu^*(y)}(T) = -\frac{1}{T}\log\mathbb{E}^{\mathbb{P}}[\frac{U^1_c(T, c^*_T)}{U^1_c(0, c^*_0)}]. \tag{5.4.2}$$

In this subsection a way to express the interest rates term structure has been proposed. But when doing this, one takes the expectation of $Y^*_t(y)$, thus some information contained in $Y^*_t(y)$ is lost. If one wants to use all the information contained in $Y^*_t(y)$, it is better to write zero coupon dynamics. This is the purpose of the next subsection.

5.4.2 Dynamic utility functions and zero coupon bond prices

In Chapter 3, it is assumed that zero coupon bond prices in an incomplete market are given by Davis prices. With this hypothesis on Davis prices, the price of a zero-coupon $B^{\nu^*(y)}(t, T)$ between t and T is, for all $0 \leq t \leq T$:

$$\begin{aligned} B^{\nu^*(y)}(t, T) &= \mathbb{E}^{\mathbb{Q}^{\nu^*(y)}}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{P}}[\exp(-\int_t^T r_s ds)\mathcal{E}(-\langle\tilde{\theta}_s, dW_s\rangle)\mathcal{E}(-\langle\nu^*_s(Y^*_s(y)), dW_s\rangle)|\mathcal{F}_t]. \end{aligned}$$

But if we want to use all the information available, we rather use the fact that $Y_t^*(y)B^{\nu^*(y)}(t,T)$ is a martingale and use the volatility of the yield curve. A similar kind of approach was presented at the end of Chapters 2 and 3 (sections 2.9.3 and 3.5), but here it is interesting to examine the differences coming from dynamic utility function.

$$Y_t^*(y)B^{\nu^*(y)}(t,T) = yB^{\nu^*(y)}(0,T)\mathcal{E}\big(\int_0^t \langle \Gamma^{\nu^*(y)}(s,T), dW_s\rangle\big).$$

At time $t = T$, because $B^{\nu^*(y)}(T,T) = 1$, we have:

$$Y_T^*(y) = yB^{\nu^*(y)}(0,T)\mathcal{E}\big(\int_0^T \langle \Gamma^{\nu^*(y)}(s,T), dW_s\rangle\big).$$

We would like to express the link between $\int_0^t r_s ds$ and the optimal strategy $\nu^*(y)$.

$$e^{-\int_0^T r_s ds}\mathcal{E}\big(-\int_0^T \langle \tilde{\theta}_s + \nu_s^*(Y_s^*(y)), dW_s\rangle\big) = B^{\nu^*}(0,T)\mathcal{E}\big(\int_0^T \langle \Gamma^{\nu^*(y)}(s,T), dW_s\rangle\big).$$

The volatility vector can be separed into two terms: one on the space of the minimal risk premium and of tradable assets, and one on the orthogonal space. Then, using the decomposition of the volatility on $\mathcal{K}$ and $\mathcal{K}^\perp$:

$$\begin{aligned}
&\mathcal{E}\big(\int_0^T \langle \Gamma^{\nu^*(y)}(s,T), dW_s\rangle\big)\\
&= \mathcal{E}\big(\int_0^T \langle \Gamma^{\nu^*(y),\sigma}(s,T), dW_s\rangle\big)\mathcal{E}\big(\int_0^T \langle \Gamma^{\nu^*(y),\perp}(s,T), dW_s\rangle\big).
\end{aligned}$$

Thus, the expression of $\int_0^T r_s ds$ can be decomposed in two parts, one involving terms of $\mathcal{K}$ and one involving terms of $\mathcal{K}^\perp$ (more precisely the optimal dual process $\nu^*(y)$ and the part of the the zero-coupon bond volatility vector which lies in $\mathcal{K}^\perp$, that is $\Gamma^{\nu^*(y),\perp}(s,T)$).

$$\begin{aligned}
&-\int_0^T r_s ds\\
&= \log B^{\nu^*(y)}(0,T) + \int_0^T \langle \Gamma^{\nu^*(y),\sigma}(s,T), dW_s\rangle - \frac{1}{2}\int_0^T ||\Gamma^{\nu^*,\sigma}(s,T)||^2 ds\\
&-\int_0^T \langle \tilde{\theta}_s, dW_s\rangle + \frac{1}{2}\int_0^T ||\tilde{\theta}_s||^2 ds\\
&+\int_0^T \langle \Gamma^{\nu^*(y),\perp}(s,T) - \nu_s(Y_s^*(y)), dW_s\rangle - \frac{1}{2}\int_0^T (||\Gamma^{\nu^*(y),\perp}(s,T)||^2 - ||\nu_s(Y_s^*(y))||^2) ds.
\end{aligned}$$

Further analysis of this equation gives more information on the structure on the zero-coupon bond volatility vector $\Gamma^{\nu^*(y)}(s,T)$.

To conclude this section, we present an HJM-like equation expressing the dynamics of $\int_0^t r_s ds$. The process $Y_t^*(y)B^{\nu^*,y}(t,T)$ is a $\mathbb{P}$-martingale. Let us define the probability measure $\mathbb{Q}^*$ such that:

$$\frac{d\mathbb{Q}^{*,y}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}(\int_0^t \langle \tilde{\theta}_s + \nu_s^*(Y_s^*(y)), dW_s \rangle),$$

and $(W_t^{*,y})_{(0\leq t\leq T)}$ such that $W_t^{*,y} = W_t - \int_0^t \tilde{\theta}_s + \nu_s^*(Y_s^*(y))ds$ is a standard $\mathbb{Q}^*$-Brownian motion.
The process $\exp(-\int_0^t r_s ds)B^{\nu^*,y}(t,T)$ is a $\mathbb{Q}^{*,y}$ martingale, which can be expressed as:

$$\exp(-\int_0^t r_s ds)B^{\nu^*,y}(t,T) = B^{\nu^*,y}(0,T)\mathcal{E}(\int_0^t \langle \Gamma^{*,y}(s,t), dW_s^{*,y} \rangle)$$

This gives the following HJM-like equation, in terms of forward rates in this context:

$$-\int_0^t r_s ds = -\int_0^t f^{*,y}(0,s)ds + \int_0^t \langle \Gamma^{*,y}(s,t), dW_s^* \rangle - \frac{1}{2}\int_0^t ||\Gamma^{*,y}(s,t)||^2 ds$$

5.5 Dynamic power utility functions

Similarly to Chapter 3, the consistent dynamic power utility function plays a special role. It is possible to express its form more precisely and to derive further results concerning the yield curve. It has been shown in [KM10b] that consistent dynamic power utility function are of the following form:

$$U^\alpha(t,x) := Z_t^{(\alpha)} \frac{x^\alpha}{\alpha},$$

where the risk aversion coefficient α satisfies $0 < \alpha < 1$. The process $Z_t^{(\alpha)}$ depends on α, it is a semi-martingale chosen in order to satisfy the consistency property of Definition 5.1.2.
Let us consider two coefficient $0 < \alpha < 1$ and $0 < \eta < 1$. The following functions are a dual dynamic utility pair:

$$U_2^\alpha(t,x) = Z_t \frac{x^\alpha}{\alpha}, \text{ and } U_1^\alpha(t,x) = \int_0^t Z_t^c \frac{x^\alpha}{\alpha},$$

where Z_t and Z_t^c are chosen in order to ensure the consistence property.
We construct mixtures of utility functions. Let U^α be a dynamic power

utility function, with α distributed according to the probability distribution μ^α. Let $Y_t^{*,\alpha}(y)$ be the optimal state price density process of the dynamic power utility function U^α:

$$Y_t^{*,\alpha}(y) = yY_t^{*,\alpha}(1) = y\exp(-\int_0^t r_s ds)\mathcal{E}(-\int_0^t \langle \nu_s^*(Y_s^{*,\alpha}(y)), dW_s\rangle)\mathcal{E}(-\int_0^t \langle \tilde{\theta}_s, dW_s\rangle),$$

where we recall that $\nu^*(.)$ and $\tilde{\theta}(.)$ are orthogonal. Then let us call U the sup convolution of these utility function. The corresponding state price density process is:

$$Y_t^*(y) = \int Y_t^{*,\alpha}(y^\alpha)\mu(d\alpha)$$

The corresponding yield curve involves the term:

$$\log \mathbb{E}^{\mathbb{P}}[\frac{1}{y}\int Y_t^{*,\alpha}(y^\alpha)\mu(d\alpha)] = \log \int \mathbb{E}^{\mathbb{P}}[\frac{1}{y}Y_t^{*,\alpha}(y^\alpha)]\mu(d\alpha).$$

This gives different forms of the yield curve. Introducing mixtures of state price density processes gives more flexibility, more degrees of freedom in the yield curve. This may be a possibility for future work.

Bibliography

[AC96] K.J. Arrow and W. Cline. *Intertemporal Equity, Discouting, and Economic Efficiency*, pages 129–144. Cambridge University Press, 1996.

[Arr95] K.J. Arrow. Effet de serre et actualisation. *Revue de l'Energie*, (471):631–636, 1995.

[Art97] Philippe Artzner. *On the numeraire portfolio, Mathematics of derivative securities*, volume 15, pages 53–58. Publ. Newton Inst., 1997.

[BBK+09] Pauline Barrieu, Harry Bensusan, Nicole El Karoui, Caroline Hillairet, Stéphane Loisel, Claudia Ravanelli, and Yahia Salhi. Understanding, modeling and managing longevity risk: Key issues and main challenges. Preprint, 2009.

[BBP97] I. Bajeux-Besnainou and R. Portait. The numeraire portfolio: a new perspective on financial theory. *The European Journal of Finance*, 3:291–309, 1997.

[Bec01] Dirk Becherer. The numeraire portfolio for unbounded semimartingales. *Finance Stoch.*, 5(3):327–341, 2001.

[BF89] Olivier Blanchard and Stanley Fischer. *Lectures on Macroeconomics*. MIT Press, 1989.

[BF01] D. Brigo and F.Mercurio. *Interest Rate Models: Theory and Practice*. Springer, 2001.

[Bre89] D.T. Breeden. An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Theory of Valuation: Frontiers of Modern Financial Theory*, 7(53), 1989.

[BRT07] F.P.Y.S. Berrier, L.C.G Rogers, and M.R. Tehranchi. A characterization of forward utility functions. Technical report, 2007.

[BS08] Wolfgang Buchholz and Jan Schumacher. Discounting the long-distant future: A simple explanation for the weitzman-gollier-puzzle. CESifo Working Paper Series CESifo Working Paper No. 2357, CESifo Group Munich, 2008.

[BT08] F.P.Y.S. Berrier and M.R. Tehranchi. Forward utility of investment and consumption. Technical report, 2008.

[CH89] J. Cox and C.F. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econom. Theory*, 49:33–38, 1989.

[CK92] J. Cvitanic and I. Karatzas. Convex duality in constrained portfolio optimization. *Annals of Applied Probability*, 2:767–818, 1992.

[Dav98] Mark H.A. Davis. Option pricing in incomplete markets. In S.R. Pliska, editor, *Mathematics of Derivative Securities*, pages 216–226. M.A.H. Dempster and S.R. Pliska, cambridge university press edition, 1998.

[DE92a] Darell Duffie and Larry G. Epstein. Asset pricing with stochastic differential utility. *The Review of Financial Studies*, 5(3):411–436, 1992.

[DE92b] Darell Duffie and Larry G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–394, March 1992.

[DJ98] Rose-Anne Dana and Monique Jeanblanc. *Marchés financiers en temps continu, Valorisation et Equilibre*. Economica, 1998.

[DM75] C. Dellacherie and P.-A. Meyer. *Probablités et Potentiel*, volume Chapitres I à IV. Hermann, Paris, 1975.

[EGR95] Nicole ElKaroui, Hélyette Geman, and Jean-Charles Rochet. Changes of numeraire, changes of probability measure and option pricing. *J. Appl. Prob.*, 32(2):443–458, 1995.

[EJJ10] Nicole ElKaroui, Monique Jeanblanc, and Ying Jiao. What happens after a default: The conditional density approach. *Stochastic Processes and their Applications*, 120:1011–1032, 2010.

[EJY00] R.J. Elliott, Monique Jeanblanc, and Marc Yor. On models of default risk. *Mathematical Finance*, 10:179–195, 2000.

[Eke] Ivar Ekeland. Long-term policy making. .

[ElK] Nicole ElKaroui. *Coverture des risques dans les marchés financiers.*

[EM10] Nicole ElKaroui and Mohamed Mrad. Consistent stochastic utilities: Approach by stochastic flows. *Preprint*, 2010.

[EQ95] Nicole ElKaroui and Marie-Claire Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33(1):29–66, January 1995.

[Gol] Christian Gollier. What is the socially efficient level of the long-term discount rate? .

[Gol02] Christian Gollier. Time horizon and the discount rate. *Journal of Economic Theory*, 107(2):463–476, December 2002.

[Gol06] Christian Gollier. An evaluation of sten's report on the economics of climate change. Technical Report 464, IDEI Working Paper, 2006.

[Gol07a] Christian Gollier. Comment intégrer le risque dans le calcul économique? *Revue d'économie politique*, 117(2):209–223, 2007.

[Gol07b] Christian Gollier. The consumption-based determinants of the term structure of discount rates. *Mathematics and Financial Economics*, 1(2):81–101, July 2007.

[Gol08] Christian Gollier. Managing long-term risks. 2008.

[Gol09a] Christian Gollier. Ecological discounting. IDEI Working Papers 524, Institut d'Économie Industrielle (IDEI), Toulouse, July 2009.

[Gol09b] Christian Gollier. Expected net present value, expected net future value and the ramsey rule. Technical Report 557, IDEI Working Paper, June 2009.

[Gol09c] Christian Gollier. Should we discount the far-distant future at its lowest possible rate? *Economics: the Open Access, Open-Assessment E-Journal*, 3(2009-25), June 2009.

[GP98] A. Grorud and M. Pontier. Insider trading in a continuous time market model,. *International Journal of Theorical and Applied Finance*, 1:331–347, 1998.

[Gue04] Roger Guesnerie. Calcul économique et développement durable. Technical Report 2004-02, DELTA Working Papers, 2004.

[HDH10] Stéphane Hallegatte, Patrice Dumas, and Jean-Charles Hourcade. A note on the economic cost of climate change and the rationale to limit it below 2 °c. Policy Research Working Paper 5179, The World Bank, January 2010.

[HG07] Cameron Hepburn and Ben Groom. Gamma discounting and expected net future value. *Journal of Environmental Economics and Management*, 53(1):99–109, January 2007.

[Hil04] Caroline Hillairet. *Equilibres sur un marché financier avec asymétrie d'information et discontinuité des prix.* PhD thesis, 2004.

[Hil05] Caroline Hillairet. Existence of an equilibrium on a financial market with discontinuous prices, asymmetric information and non trivial initial sigma-fields. *Mathematical Finance*, 15(1):99–117, January 2005.

[HJ10] Caroline Hillairet and Ying Jiao. Information asymmetry in pricing of credit derivatives. Preprint, February 2010.

[HJM98] David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105, January 1998.

[HK04] Julien Hugonnier and Dmitry Kramkov. Optimal investment with random endowments in incomplete markets. *Annals of Applied Probability*, 14(845), 2004.

[HP91] H. He and N.D. Pearson. Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case. *Mathematical Finance*, 1:1–10, 1991.

[HS09] Lars Peter Hansen and José A. Scheinkman. Long-term risk: An operator approach. *Econometrica*, 77(1):177–234, January 2009.

[Jac79] Jean Jacod. *Calcul stochastique et problèmes de martingales.* Number Lecture Notes 714. Springer-Verlag, 1979.

[Jac85] Jean Jacod. *Grossisement Initial, hypothèse (H') et Théorème de Girsanov*, volume Lecture Notes 1118. Springer-Verlag, 1985.

[Jea] Monique Jeanblanc. Credit risk modeling. Notes de Cours.

[JMN10] Elyès Jouini, Jean-Michel Marin, and Clotilde Napp. Discounting and divergence of opinion. *Journal of Economic Theory*, 145:830–859, 2010.

[JN10] Elyès Jouini and Clotilde Napp. Unbiased disagreement in financial markets waves of pessimism and the risk-return trade-off. Cass Business School, 2010, April 2010.

[Joua] Elyès Jouini. Behavioral biases and representative agent (or the brain as a central planner with heterogenous doers). IHP 2010.

[Joub] Elyès Jouini. Long term discounting and heterogeneous beliefs. IHP 2010.

[Jou10] Elyès Jouini. Discount rates: the equilibrium approach. IHP 2010, 2010.

[JR00] Monique Jeanblanc and Marek Rutkowski. Modelling of default risk: Mathematical tools. 2000.

[Kel56] J.R. Kelly. A new interpretation of information rate. *Bell Syst. Techn. J.*, 35:917–926, 1956.

[KLSX91] I. Karatzas, J.P. Lehoczky, S.E. Shreve, and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control and Optimization*, 29:702–730, 1991.

[KM10a] Nicole El Karoui and Mohamed Mrad. Drift and volatility characterization of progressive utility: Stochastic pde's and stochastic flows method. Preprint, April 2010.

[KM10b] Nicole El Karoui and Mohamed Mrad. Mixture of consistent stochastic utilities and a priori randomness. Preprint, May 2010.

[KS91] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 1991.

[KS98] T. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer, 1998.

[KS99] Dmitry Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9:904–950, 1999.

[KS03] Dmitry Kramkov and Walter Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, 13(4):1504–1516, 2003.

[LH] Franck Lecocq and Jean-Charles Hourcade. Le taux d'actualisation contre le principe de précaution? Leçons à partir du cas des politiques climatiques.

[Lon90] J.B. Long. The numeraire portfolio. *J. Financial Economics*, 26:29–69, 1990.

[LQ03] Ali Lazrak and Marie-Claire Quenez. A generalized stochastic differential utility. *Math. Operations Research*, 28(1):154–180, February 2003.

[LZ04] Ali Lazrak and Fernando Zapatero. Efficient consumption set under recursive utility and unknown beliefs. *Journal of Mathematical Economics*, 40:207–226, 2004.

[Man95] A. Manne. The rate of time preference: implications for the green house debate. *Energy Policy*, 23(4/5):391–394, 1995.

[Mar08] Ian Martin. The valuation of long-dated assets. November 2008.

[Mer69] Robert C. Merton. Lifetime portfolio selection under uncertainty: the continuous time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

[Mer71] Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *J. Econom. Theory*, 3(4):373–413, 1971.

[MR00] Marek Musiela and Marek Rutkowski. *Martingale Methods in Financial Modelling*. Springer, 2000.

[Mra09] Mohamed Mrad. *Utilités progressives dynamiques*. PhD thesis, Thèse de doctorat, Ecole Polytechnique, October 2009.

[PH06] E. Platen and D. Heath. *A Benchmark Approach to Quantitative Finance*. Springer Finance, Springer-Verlag, Berlin, 2006.

[Pha07] Huyên Pham. *Optimisation en contrôme stochastique appliqués à la finance*. Springer, July 2007.

[Pla04a] E. Platen. Intraday empirical. Technical Report 125, Quantitative Finance Research Centre, University of Technology Sydney, May 2004.

[Pla04b] Eckhard Platen. Diversified portfolios with jumps in a benchmark framework. Technical report, University of Technology Sydney, June 2004.

[Pla04c] Eckhard Platen. Modeling the volatility and expected value of a diversified world index. *International Journal of Theorical and Applied Finance*, 7(4):511–529, 2004.

[Pla06] Eckhard Platen. A benchmark approach to finance. *Mathematical Finance*, 16(1):131–151, 2006.

[Pla09] Eckhard Platen. Real world pricing of long term contracts. Working Paper 262, University of Technology Sydney, November 2009.

[Pli86] S.R. Pliska. A stochastic calculus model of continuous trading: optimal portfolios. *Math. Operations Research*, 11:371–382, 1986.

[PR09] Eckhard Platen and Renata Rendek. Simulation of diversified portfolios in a continuous financial market. Working Paper 264, University of Technology Sydney, December 2009.

[Sam69] Paul A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246, 1969.

[Ste] Stern. Stern review on climate change. .

[Wei98] Martin L. Weitzman. Why the far-distant future should be discounted at its lowest possible rate. *Journal of Environmental Economics and Management*, 36(3):201–208, November 1998.

[Wei07] Martin L. Weitzman. A review of the the stern review on the economics of climate change. *Journal of Economic Litterature*, 45:703–724, September 2007.

[Zar94] T. Zariphopoulou. Consumption-investment models with constraints. *SIAM J. Control and Optimization*, 32(1):59–85, 1994.

Part II

Stochastic Analysis of determinantal point processes

Introduction

Cette partie a pour objet l'analyse stochastique des processus ponctuels déterminantaux. La motivation sous-jacente était l'étude de processus ponctuels ayant une riche structure de corrélation, pour pouvoir ensuite modéliser des évènements corrélés ou des configurations de points dans l'espace.

Dans cette partie nous allons donc étudier des processus ponctuels, c'est à dire des configurations aléatoires de points situés sur une droite ou dans l'espace. Une multitude de phénomènes physique ou de la vie courante peut être modélisée par ce genre de processus: les dates d'occurrence de certains évènements (par exemple en finance des dates de faillite d'entreprises), les temps d'arrivées de clients dans une file d'attente, les positions de particules dans l'espace, ou les arrangements d'étoiles dans une galaxie.
Pour mener à bien cette modélisation il est nécessaire d'avoir accès à des familles de processus ponctuels pour lesquels on connaît suffisamment les propriétés mathématiques et qui soient aussi suffisamment riches pour pouvoir bien capturer les propriétés qualitatives des processus qu'on cherche à modéliser.

Parmi les différentes familles de processus connus, les processus de Poisson sont les mieux étudiés et ceux pour lesquels les calculs sont relativement faciles à faire. Cependant, ils sont caractérisés par une propriété d' indépendance : le nombre de points du processus qui tombe dans des ensembles A et B disjoints sont deux variables aléatoires indépendantes. Mais lorsqu'on fait de la modélisation, cette hypothèse d'indépendance peut être justifiée dans certains cas, mais bien sûr pas tous.

En effet, cette hypothèse d'indépendance ne tient pas compte de la corrélation qu'il peut y avoir entre les différents points du processus.

Par exemple, si on cherche à modéliser la propagation d'une maladie contagieuse dans une région, et qu'un point de notre processus ponctuel représente la localisation d'un cas de cette maladie, alors il est très probable

qu'il y ait d'autres points au voisinage de ce point.
Au contraire, si on cherche modéliser les positions de particules de même charge, celles-ci vont avoir tendance à se repousser. Si une particule est située en un point donné, alors il est peu probable qu'il y en ait d'autres dans son voisinage.
En finance, si on cherche à modéliser des dates de défauts d'entreprises, la situation est plus complexe car de nombreux paramètres économiques interviennent. Cependant, dans le cas d'une crise systémiques, si beaucoup de défauts d'entreprises surviennent en même temps, on se rapproche donc du premier exemple, celui où les points (ici ces points représentent les dates de défaut) vont être proches les uns des autres.

Ainsi en pratique, il y a principalement deux manières de s'éloigner de l'hypothèse d'indépendance. Soit on constate plutôt de l'attraction entre les points (corrélation positive), soit plutôt de la répulsion (corrélation négative). Une question naturelle qui se pose alors naturellement est : y-a-t-il des processus qui génèrent de tels comportements? Et peut-on les utiliser pour la modélisation?
Ainsi, on cherche à étudier des configurations aléatoires de points, dans lesquelles les points se repoussent ou s'attirent, respectivement, ainsi ces processus ponctuels sont très loin de la situation de non-corrélation rencontrée pour les processus ponctuels de Poisson.

Une famille bien connue de processus ponctuels donne lieu au cas où les points s'attirent, il s'agit des processus de Cox. Plus précisément, soit $X(.)$ un processus continu à valeurs dans $\mathbb{R}^+$. On considère un processus de Poisson d'intensité $\mu(A) = \int_A X(t) d\lambda(t)$, où λ est une mesure de référence sur l'espace considéré. Ce processus est un processus de Cox.

Intuitivement on peut constater que là où X prend de grandes valeurs, l'intensité aussi, on a alors une plus grande probabilité d'avoir un point. Alors X prend aussi de grandes valeurs aux voisinage de ce point. On a aussi une plus grande probabilité d'avoir d'autres points au voisinage de ce point. On observe donc des amas de points. Ainsi les points de ce processus vont avoir tendance à s'attirer.

Cela nous amène à la question suivante: existe-t-il au contraire des familles de processus ponctuels qui génèrent un phénomène de répulsion entre leurs points?

Les processus ponctuels déterminantaux sont un tel exemple de processus, et c'est l'étude de leurs propriétés qui va nous occuper dans cette partie. Nous les noterons aussi DPP, abbréviation de Determinantal Point Processes.

Les processus ponctuels déterminantaux ont été d'abord introduits par Macchi [31] pour représenter des configurations de particules, par exemple des électrons. Pour les désigner, le terme de *fermion point processes* a d'abord été utilisé. Le terme *déterminantal* a été utilisé entre autres par [11], et cette expression est désormais standard.

Les processsus déterminantaux ont une propriété de répulsion qui provient de leur définition. Pour se rendre compte cette propriété de répulsion, il suffit de les comparer aux processus de Poisson. Dans les figures suivantes on représente un processus ponctuel de Poisson dans le plan et un processus déterminantal dans le plan ayant le même nombre de points en moyenne. On peut ainsi constater qualitativement la différence entre les processus déterminantaux (phénomène de répulsion) et les processus de Poisson (absence de corrélation). Des éléments concernant la simulation des processus déterminantaux seront donnés dans le chapitre 6.

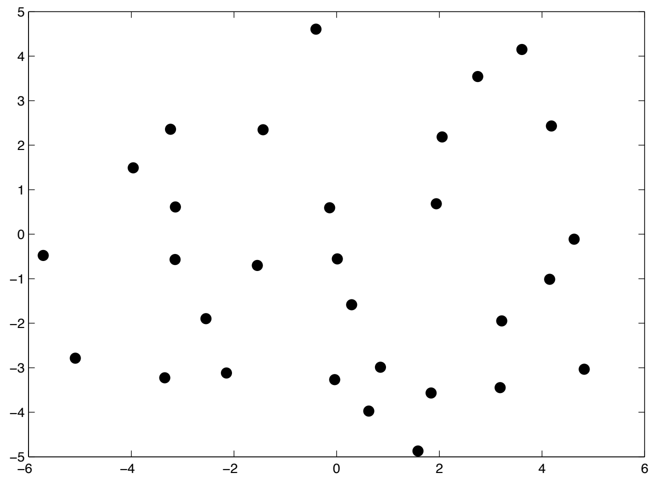


Figure 5.1: Processus déterminantal

Dans la suite nous étudierons les processus déterminantaux dans $\mathbb{R}^d$. Dans les exemples, nous insisterons sur les processus déterminantaux sur la droite réelle et dans le plan (surtout pour les simulations).

Nous allons considérer $\Lambda \in \mathbb{R}^d$ et λ est une mesure de référence sur Λ (par exemple la mesure de Lebesgue dans le cas le plus simple).

Pour motiver la définition des processus déterminantaux et illustrer le fait qu'ils apparaissent en physique, nous mentionnons l'exemple suivant qui vient

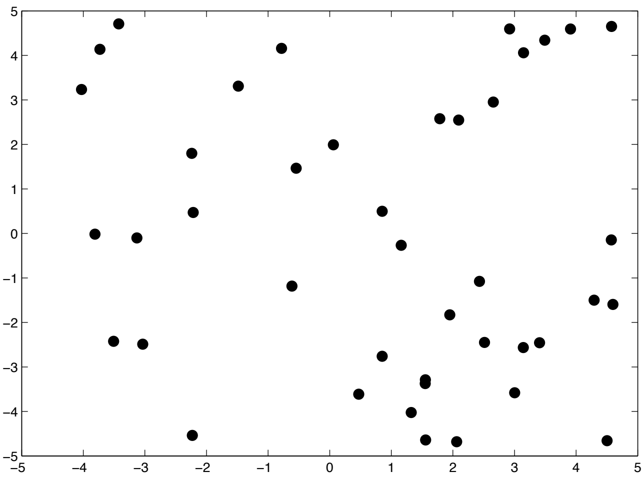


Figure 5.2: Processus de Poisson

de [24]. Nous rappelons qu'en mécanique quantique, une quantité physique, comme la position d'un éléctron est représentée par une fonction à valeurs complexes (la fonction d'onde) ϕ telle que $\int |\phi|^2 = 1$. Alors $|\phi|^2$ est la densité de probabilité de la position en question. Maintenant considérons n fonctions d'onde individuelles $\phi_1, \ldots, \phi_n$ sur Λ. Pour construire une fonction d'onde $\phi_1 \otimes \ldots \otimes \phi_n$ à n particules les physiciens tiennent compte du fait que ces particules ne peuvent être distinguées entre elles et qu'elles se repoussent, et ils rendent anti-symmétrique la nouvelle fonction d'onde, c'est-à-dire:

$$\frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} sgn(\pi) \prod_{i=1}^{n} \phi_{\pi_i}(x_i) = \frac{1}{\sqrt{n!}} \det((\phi_j(x_i)))_{i,j \leq n}.$$

Si les ϕ_i forment une famille orthogonale, alors le produit scalaire:

$$\langle \prod_{i=1}^{n} \phi_{\pi_i}(x_i), \prod_{i=1}^{n} \bar{\phi_{\sigma_i}}(x_i) \rangle,$$

est nul sauf si $\pi = \sigma$. On peut alors vérifier que cette fonction d'onde est une densité de probabilité sur Λ^n et s'écrit aussi:

$$\frac{1}{n!} \det((\phi_j(x_i))) \det((\bar{\phi_i}(x_j))) = \frac{1}{n!} \det((K(x_i, x_j)))_{i,j \leq n},$$

où $K(x,y) = \sum_{i=1}^{n} \phi_i(x) \bar{\phi_i}(y)$. Ainsi, les fonctions d'ondes de n électrons s'expriment comme des densités de probabilités faisant intervenir le déterminant d'une matrice (que nous détaillerons plus tard). Cette densité de probabilité s'annule quand $x_i = x_j$ pour $i \neq j$, ce qui indique que les points ont tendance à se repousser et que deux points ne peuvent pas se trouver au même endroit.

Depuis qu'ils ont été introduits par Macchi [31], les processus ponctuels déterminantaux ont été étudiés avec différents points de vue que nous mentionnons ici. Spohn [41] a étudié la dynamique du modèle de Dyson's, un modèle de particules en interaction sur la droite réelle. La mesure invariante qui apparaît est un processus ponctuel déterminantal qui dépend d'un noyau sinus.
Plus récemment, ces processus ont été considérés aussi sous un autre angle car ils apparaissent aussi très naturellement dans l'étude du spectre des matrices aléatoires [27, 39]. En effet les valeurs propres de certaines matrices aléatoires (hermitiennes, unitaires, orthogonales,...) sont distribuées comme un processus déterminantal. De même, les valeurs propres des matrices de l'ensemble de Ginibre forment également un processus déterminantal.
Les processus ponctuels déterminantaux apparaissent aussi dans les graphes aléatoires, voir Johansson [25, 26, 27], et les fonctions gaussiennes analytiques (voir [23]).
Récemment aussi, Soshnikov [39] a établi des théorèmes d'existence pour les processus ponctuels déterminantaux, a donné des exemples des processes déterminantaux apparaissant à la fois en mathématiques et en physique, des résultats concernants les processus déterminantaux ayant un noyau invariant par translation, en particulier des théorèmes centraux limites.
Enfin, on peut aussi trouver une discussion détaillée sur les propriétés probabilistes des processus ponctuels déterminantaux ainsi que leurs applications dans de nombreuses références, parmi lesquelles nous pouvons citer : see [39, 30, 27, 23, 40, 13, 24, 37, 21, 12, 29].

Macchi [31] a aussi identifié une autre famille de processus, qui présente à la fois des similarités et des différences avec les déterminantaux. Il s'agit des processus ponctuels permanentaux. Ils peuvent modéliser des configurations de bosons (par exemple photons) dans l'espace. Dans ce cas au contraire, il y a un phénomène d'attraction entre les points. Nous verrons plus loin comment ces deux familles de points (déterminantaux et permanentaux peuvent être réunies.

Nous commençons ici à décrire plus en détail les propriétés des processus déterminantaux. Ces processus sont définis par leurs fonctions de corrélation. La fonction de corrélation $\rho(x_1, ..., x_n)$ représente la probabilité d'avoir n

particules indentiques au voisinage de x_1,...,x_n. Pour plus de détails voir le livre de Daley et Vere-Jones [17]. Pour un processus déterminantal cette fonction de corrélation prend la forme d'un certain déterminant.
Plus précisément, la fonction de corrélation $\rho(x_1, ..., x_n)$ est égale au déterminant d'une matrice obtenue en prenant les valeurs du noyau d'un opérateur intégral aux points $(x_1, \ldots, x_n)$. C'est de ce déterminant que provient le nom de processus déterminantaux.
Comme ces processus font apparaître les corrélations de façon explicite, on peut penser qu'ils peuvent être utiles pour modéliser des situations dans lesquelles la dépendence ou la corrélation entre les points est cruciale.
Constatons aussi que pour les processus permanentaux précédemment mentionés, leur fonction de corrélation s'exprime aussi de façon simple (avec un permanent à la place d'un déterminant).

Shirai and Takahashi [37] ont proposé une présentation unifiée des processus déterminataux et permanentaux en introduisant une classe plus générale de processus, les processus α-déterminantaux. Un processus α-déterminantal dépend du noyau K d'un certain opérateur intégral (vérifiant certaines conditions que nous préciserons par la suite), d'un réel α, qui mesure la force de la corrélation entre les particules et d'une mesure de référence λ. Dans [37], ils ont établi que la transformée de Laplace de ces processus ponctules est égale à la puissance $-\frac{1}{\alpha}$ d'un certain déterminant de Fredholm . Le paramètre α est à valeurs dans $\{2/m, m \in \mathbb{R}\} \cup \{-1/m, m \in \mathbb{R}\}$ et il représente est une indication de la structure de corrélation du processus. Quand $\alpha < 0$ il y a un phénomène de répulsion et quand $\alpha > 0$, il y a un phénomène d'attraction. Les processus déterminantaux et permanentaux mentionnés précédemment sont des cas particuliers des processus α-déterminantaux (respectivement pour $\alpha = -1$ et $\alpha = 1$).
Mais mis à part les cas $\alpha = 1$ et $\alpha = -1$, les autres valeurs de α pour les α-déterminantaux n'ont pas d'interprétation physique (contrairement à l'analogie fermions correspondant à $\alpha = -1$ et bosons correspondant à $\alpha = 1$, les processus α-déterminantaux ne correpondent pas à une famille de particules).

A vrai dire l'existence de processus α-déterminantaux n'est même pas garantie pour toutes les valeurs de α (voir [39]). L'existence a été démontrée pour $\alpha = \pm 1$ (processus déterminantaux et permanentaux). Les processus α-déterminantaux pour les autres valeurs de α admissibles sont construits comme des superposition des "briques" que sont les processus permanentaux et déterminantaux. L'existence de processus α-déterminantaux pour d'autres valeurs de α est une question ouverte.

Cette famille de processus a certes une riche structure de corrélation mais elle dépend très fortement du noyau K et de ses valeurs, ce qui rend les calculs complexes et une description de ces processus assez difficile. Par ailleurs on se rend compte que nombre de ses propriétés mathématiques sont méconnues.
Ainsi notre objectif est de développer certaines propriétés stochastiques de ces processus. Plus précisément, leurs propriétés en tant que processus stochastiques, comment on peut les perturber, que penser de leur les simuler sont des questions qui ont été peu ou pas abordés dans la littérature existante.
Comme ces processus dépendent très fortement de la corrélation, il nous semble qu'en dimension 1 (ils sont alors indexés par le temps) un calcul similaire au calcul d'Itô a peu de chance d'aboutir. En effet, du fait de la forte structure de corrélation, chaque point du processus (représentant par exemple un évènement) dépend de tous les évènements futurs. On s'imagine mal avoir une formule fermée pour un compensateur.

Dans l'esprit de [46], nous nous souhaitons alors établir un calcul différentiel pour les processus déterminantaux et nous nous intéressons plus particulièrement au calcul de Malliavin.
A ce jour, le calcul de Malliavin a été développé pour d'autres processus ponctuels. C'est le cas pour les processus de Poisson ([3, 6, 7, 10, 18, 35], [34], [33]), pour les processus de Gibbs [4], les processus ponctuels marqués [2], les processus de Poisson filtrés [18], les cluster processes [9] et les processus de Lévy [5, 19].
Ce n'est pas le cas en revanche pour les processus ponctuels déterminantaux et c'est ce que nous allons faire, en définissant une dérivée en perturbant des configurations, comme dans Albeverio et al. [3].

Plan La motivation d'origine de ce travail était d'étudier plus en détail la distribution de ces processus qui ont une riche structure de corrélation, en particulier ces comportements d'attraction et de répulsion et de voir s'ils pourraient être utilisés pour la modélisation.
Ainsi dans le chapitre 6 nous présentons d'abord les propriétés des processus déterminantaux, permanentaux, et α-déterminantaux. Pour cela nous nous appuyons essentiellement sur le travail de Shirai et Takahashi [37], et Hough et al. [23, 24]. Nous décrivons un certain nombre d'exemples de noyaux. Dans [37], il est établi que la transformée de Laplace de ces pro-

cessus ponctuels est égale à la puissance $-\frac{1}{\alpha}$ d'un certain déterminant de Fredholm. Ainsi, pour un processus α-déterminantal (noté ξ) caractérisé par les paramètres $(\alpha, K_\Lambda, \lambda)$, sa transformée de Laplace s'écrit:

$$\mathbb{E}\left[\exp\left(-\int f \quad \mathrm{d}\xi\right)\right] = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda[1 - e^{-f}])^{-1/\alpha},$$

où Det(.) est un déterminant de Fredholm et $K_\Lambda[1 - e^{-f}]$ est l'opérateur de noyau:

$$K_\Lambda[1 - e^{-f}](x, y) = \sqrt{1 - e^{-f(x)}} K_\Lambda(x, y) \sqrt{1 - e^{-f(y)}}.$$

Nous rappelons les autres quantités permettant de caractériser les processus déterminantaux. Par exemple, leurs fonctions de corrélation s'expriment comme $\rho(x_1, \dots, x_n) = \det_\alpha(K_\Lambda(x_i, x_j))_{1 \le i,j \le n}$, où $\det_\alpha$ est la généralisation d'un déterminant usuel que nous donnerons (pour $\alpha = -1$, c'et d'ailleurs un déterminant usuel).
D'autre part, l'opérateur $J_{\Lambda,\alpha}$ est défini en fonction de l'opérateur K_Λ par:

$$J_{\Lambda,\alpha} = (I + \alpha K_\Lambda)^{-1} K_\Lambda.$$

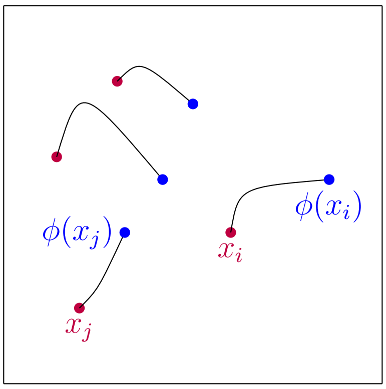
Alors les densités de Janossy du processus, qui donnent la probabilité d'avoir n points situés en $x_1, \dots, x_n$ et aucun point ailleurs s'expriment en fonction de l'opérateur $J_{\Lambda,\alpha}$. Plus précisément, pour tout $n \ge 1$, pour tout $(x_1, \dots, x_n) \in \Lambda^n$, les densités de Janossy sont définies par:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \dots, x_n) = \mathrm{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \det_\alpha(J_{\Lambda,\alpha}(x_i, x_j))_{1 \le i,j \le n},$$

et $j^0_{\Lambda,\alpha,K_\Lambda}(\emptyset) = \mathrm{Det}(I + \alpha K_\Lambda)^{-1/\alpha}$. Des travaux précédents (par exemple Shirai [36]) ont établi que toutes ces quantités sont bien positives.
Dans la suite de ce premier chapitre concernant les déterminantaux (chapitre 6), nous nous plaçons dans le cas où λ la mesure de référence est la mesure de Lebesgue sur $\mathbb{R}^d$. Ce chapitre sert d'introduction au suivant: nous présentons les résultats essentiels existants concernant les déterminantaux, et nous donnons aussi quelques propriétés que nous avons établies. Cela permet d'avoir une vue d'ensemble des déterminantaux avant de passer aux questions d'intégration par parties. Etant donné un processus déterminantal ou un permanental sur un compact Λ, nous calculons la distribution du nombre de points dans $D \subset \Lambda$, en faisant intervenir les valeurs propres de l'opérateur K restreint au sous-espace D. Inversement, étant donné deux configurations (d'un processus déterminantal) $\xi_1 \subset \xi_2$, nous donnerons la loi de ξ_2 sachant ξ_1.
Nous nous intéressons ensuite à la simulation des processus alpha-déterminantaux. Dans le cas discret, leur simulation a été étudiée par Moller.

Sur $\mathbb{R}^n$, un algorithme a été proposé par Hough et al. [23]. Cependant, il repose sur la décomposition spectrale des opérateurs K dont dépendent les déterminantaux, et qui n'est pas forcément aisée en pratique. Par conséquent, nous proposons quelques pistes pour d'autres méthodes de simulation.
Pour les processus de Poisson ($(N_t)_{t\in\mathbb{R}^+}$) ou de Cox, on a l'habitude de considérer leur intensité. Ici pour les déterminantaux, nous essayons de faire le lien avec ce cadre en calculant leurs intensités conditionnelles au sens de Daley et Vere-Jones [17] pour un processus déterminantal sur $\mathbb{R}^+$. Ceci est l'objet du dernier paragraphe de ce chapitre.

Le chapitre 7 est consacré à une formule d'intégration par parties pour les processus déterminantaux et permanentaux. Contrairement au chapitre précédent, ici on modifie la mesure de référence λ, c'est même là-dessus que reposent la plupart des preuves.
Nous avons d'abord établi un résultat de quasi-invariance : nous avons montré que si une configuration de points (d'un processus alpha-déterminantal) est perturbée par un difféomorphisme ϕ, suivant la méthode d'Albeverio et al. [3], la configuration qui en résulte est toujours un processus déterminantal (respectivement un permanental), dont la loi est absolument continue par rapport à la distribution d'origine.

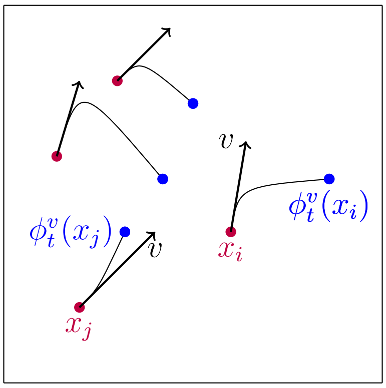


Plus précisément, on considère un processus alpha-déterminantal ξ de paramètres $(\alpha, K_\Lambda, \lambda)$, où $\alpha \in \{2/m, m \geq 1\} \cup \{-1/m, m \geq 1\}$, K_Λ est un opérateur intégral vérifiant des conditions que nous préciserons, et λ est une mesure de référence. Alors pour tout fonction f positive à support compact, la formule de quasi-invariance s'écrit:

$$\mathbb{E}[\exp(-\int f \circ \phi \, d\xi)] = \mathbb{E}[\exp(-\int f \, d\xi)\exp(\int \ln(p_\phi^\lambda) d\xi)\frac{\det_\alpha J^\phi_{\Lambda,\alpha}(\xi)}{\det_\alpha J_{\Lambda,\alpha}(\xi)}],$$

où le terme $p_\phi^\lambda = d\lambda^\phi/d\lambda$ est la densité de la mesure λ^ϕ (ou λ^ϕ est l'image de la mesure λ par ϕ) par rapport à la mesure λ. Dans cette formule, le seul terme supplémentaire par rapport à la formule de quasi-invariance pour les processus de Poisson est le quotient des deux $\det_\alpha$. Ainsi, tout au long de ce chapitre, l'importance de l'opérateur $J_{\Lambda,\alpha}$ est cruciale, ainsi que le fait de changer la mesure de référence λ.
L'étape suivante consiste à perturber la configuration par un difféomorphisme particulier ϕ_t^v, défini de façon à ce que la configuration soit perturbée le long d'un champ de vecteurs v. La formule de quasi-invariance devient:

$$\mathbb{E}[\exp(-\int f\circ\phi_t^v\ d\xi)] = \mathbb{E}[\exp(-\int f\ d\xi)\exp(\int \ln(p^\lambda_{\phi_t^v})d\xi)\frac{\det_\alpha J_{\Lambda,\alpha}^{\phi_t^v}(\xi)}{\det_\alpha J_{\Lambda,\alpha}(\xi)}].$$



On prendra v dans l'ensemble des champs de vecteurs de E (où typiquement $E=\mathbb{R}^n$) qui soient C^∞ à support compact. Alors pour une fonction $F:\chi\to\mathbb{R}$ sur l'espace des configurations, et différentiable en $\xi\in\chi$, sa dérivée le long de v s'écrit:

$$\nabla_v F(\xi) = \frac{d}{dt}F(\phi_t^v(\xi))\Big|_{t=0}.$$

Puis d'une manière analogue à [3, 8, 9], en dérivant la formule de quasi-invariance, nous obtenons une formule d'intégration par parties pour les processus déterminantaux sur l'espace des configurations.
Nous commençons par le cas des processus déterminantaux, c'est-à-dire $\alpha=-1$. Nous examinons en détails les conditions d'intégrabilité des expressions considérées. Sous ces conditions, pour F, G fonctions cylindriques, la formule d'intégration par parties s'écrit:

$$\int_\chi \nabla_v F(\xi)G(\xi)d\mu(\xi) = -\int_\chi F(\xi)\nabla_v G(\xi)d\mu(\xi)$$
$$+\int_\chi \Big(\int_E (\frac{\nabla\rho(x)}{\rho(x)}.v(x) + div(v(x)))d\xi(x) - \log\det J_{\Lambda,-1}(\xi)\Big)F(\xi)G(\xi)d\mu(\xi),$$

où ρ est la densité de la mesure λ par rapport à la mesure de Lebesgue. Il est ensuite possible d'étendre cette formule d'intégration par parties au cas des processus α-déterminantaux.

Chapter 6

Determinantal point processes

6.1 Point processes

We remind here some properties of point processes we refer to [17, 28] for more details. In the mathematical modelling of multi-component stochastic systems it is usual to describe their behaviour in terms of random configurations of points, or particles. Thus we consider a system of points moving on E, a Polish space and λ a Radon measure on $(E, \mathcal{B})$, the Borel σ-algebra on E.
By χ we denote the space of all locally finite configurations on E:

$$\chi = \{\xi \subset E : |\xi \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset E\},$$

where $|A|$ is the cardinality of a set A.
Hereafter we identify a locally finite configuration ξ, defined as a set, and the atomic measure $\sum_{x \in \xi} \delta_x$. The space χ is then endowed with the vague topology of measures and $\mathcal{B}(\chi)$ denotes the corresponding Borel σ-algebra. We note $\xi = \sum_{i, x_i \in \xi} \delta_{x_i} \in \chi$ and for any measurable nonnegative function f on E, we denote equivalently:

$$\xi \in \chi \to \langle f, \xi \rangle = \sum_{x_i \in \xi} f(x_i) = \int f d\xi.$$

We also denote by $\chi_0 = \{\alpha \in \chi, \mid \alpha(E) \mid < \infty\}$ the set of all finite configurations in χ and χ_0 is equipped with the σ-algebra $\mathcal{B}(\chi_0)$. The restriction of a configuration ξ to a compact $\Lambda \subset E$, is denoted by ξ_Λ. We introduce the set $\chi_\Lambda = \{\xi \in \chi, \xi(E \backslash \Lambda) = 0\}$. Then for any integer n, we denote by $\chi_\Lambda^{(n)} = \{\xi \in \chi, \xi(\Lambda) = n\}$, the set of all configurations in with n points in Λ. Note that we have $\chi_\Lambda = \bigcup_{n=0}^{\infty} \chi_\Lambda^{(n)}$.

Definition 6.1.1 *A random point process is a triplet* $(\chi, \mathcal{B}(\chi), \mu)$*, where* μ *is a probability measure on* $(\chi, \mathcal{B}(\chi))$*.*

Remark 6.1.1 *In the following, we denote equivalently the measure* μ *and the point process characterized by the measure* μ*.*

We denote by $\xi(\Lambda)$ the total number of points of configuration ξ in compact Λ. So $\mathbb{P}(\xi(\Lambda) = n)$ is the probability of having n points in Λ.
In the following we will restrict our study to the compact $\Lambda \subset E$, where the process ξ has a finite number of points. Hence for our observation interval Λ, the finite point processes framework described in [17] is perfectly adapted. We recall here the definition from [17] of a finite point process:

Definition 6.1.2 *A finite point process is a point process in which the numer of points is finite with probability* 1*. i) The points are located in a complete separable metric space* Λ*.*
ii) A distribution $\mathbb{P}(\xi(\Lambda) = n)$*, for all* $n \geq 0$*, is given, dtermining the total number of points in the process, with* $\sum_{n=0}^{\infty} \mathbb{P}(\xi(\Lambda) = n) = 1$*.*
iii) For each integer $n \geq 1$*, a probability distribution* $\Pi_{n,\Lambda}(.)$*, giving the joint distribution of the points of the process, conditionnally to the fact that their total number is* n*, is given on the Borel sets of* $\Lambda^{(n)} = \Lambda \times \ldots \times \Lambda$*.*

We also denote by $\pi_{n,\Lambda}(x_1, \ldots, x_n)$ the associated density, that is the joint density of $(x_1, \ldots, x_n)$ conditionally to the fact that there are n points in Λ. Notice that we are considering unordered sets of points. That is, a given configuration is a set of locations where the points are, but there is no name or label associated to a particular point. Hence there is no way to distinguish between, say, (x_1, x_2) and (x_2, x_1).
Here we are on a general domain $E \subset \mathbb{R}^d$, and at the exception of section 6.6, the quantities introduced here do not involve the order properties of the real line, although they of course hold in the case where $d = 1$ and the process is on the real line.
Every measure μ on the configuration space χ can be characterized by its Laplace function, that is for any measurable non-negative function f on E:

$$f \longmapsto \mathbb{E}_\mu[e^{-\int f \, \mathrm{d}\xi}] = \int_\chi e^{-\int f \, \mathrm{d}\xi} \, \mathrm{d}\mu(\xi).$$

For instance, let μ_σ denote the Poisson measure on $(\chi, \mathcal{B}(\chi))$ with intensity measure σ. Then its Laplace transform is, for any measurable non-negative function f with compact support:

$$\int_\chi e^{-\int f \, \mathrm{d}\xi} \, \mathrm{d}\mu_\sigma(\xi) = \exp\left(\int_E (1 - e^{-f(x)}) \, \mathrm{d}\sigma(x)\right) \tag{6.1.1}$$

We note $\xi(D)$ the number of points in D, where D is a subset of Λ. Another way to describe the distribution of a point process is to give the probabilities $\mathbb{P}(|\xi_{\Lambda_k}| = n_k,\ 1 \le k \le n)$ for any n and any mutually disjoints Borel subsets of Λ, $\Lambda_1, \cdots, \Lambda_k$, $1 \le k \le n$. For instance, the Poisson measure μ_σ with intensity measure σ can be defined in this way as:

$$\mathbb{P}(|\xi_{\Lambda_k}| = n_k,\ 1 \le k \le n) = \prod_{k=1}^{n} e^{-\sigma(\Lambda_k)} \frac{\sigma(\Lambda_k)^{n_k}}{n_k!}.$$

But in many cases, specifying the joint distribution of the $\xi(D)$'s is not simple. Then the distribution of a point process can be described by its correlation function instead.

Definition 6.1.3 *A locally integrable function $\rho_n : E^n \to \mathbb{R}_+$ is the n-point correlation function of μ if for any disjoint bounded Borel subsets $\Lambda_1, \ldots, \Lambda_m$ of E and $n_i \in \mathbb{N}$, $\sum_{i=1}^m n_i = n$:*

$$\mathbb{E}_\mu \left[\prod_{i=1}^{m} \frac{|\xi_{\Lambda_i}|!}{(|\xi_{\Lambda_i}| - n_i)!} \right] = \int_{\Lambda_1^{n_1} \times \ldots \times \Lambda_m^{n_m}} \rho_n(x_1, \cdots, x_n)\ d\lambda(x_1) \ldots\ d\lambda(x_n),$$

where $\mathbb{E}_\mu$ denotes the expectation relatively to μ and λ is a reference measure on E.

For example if $m = 1$ and $n_1 = n$, then the formula becomes:

$$\begin{aligned} \mathbb{E}_\mu \left[\frac{|\xi_\Lambda|!}{(|\xi_\Lambda| - n)!} \right] &= \mathbb{E}_\mu \left[|\xi_\Lambda| \left(|\xi_\Lambda| - 1 \right) \ldots \left(|\xi_\Lambda| - n + 1 \right) \right] \\ &= \int_{\Lambda^n} \rho_n(x_1, \cdots, x_n)\ \mathrm{d}\lambda(x_1) \ldots\ \mathrm{d}\lambda(x_n). \end{aligned}$$

We recognize here the n-th factorial moment of $|\xi_\Lambda|$. In particular:

$$\mathbb{E}_\mu[|\xi_\Lambda|] = \int_\Lambda \rho_1(x)\ \mathrm{d}\lambda(x),$$

i.e., ρ_1 is the mean density of particles ([39]). More generally, the function $\rho_n(x_1, \ldots, x_n)$ has the following interpretation: $\rho_n(x_1, \ldots, x_n)\ \mathrm{d}\lambda(x_1) \ldots\ \mathrm{d}\lambda(x_n)$ is approximately the probability to find a particle in each one of the $[x_i, x_i + \ \mathrm{d}x_i]$, $i = 1 \ldots n$.

A third way to define a point process proceeds via the Janossy densities. Denote by $\pi_{n,\Lambda}(x_1, \cdots, x_n)$ the density (assumed to exist) with respect to $\lambda^{\otimes n}$ of the joint distribution of $(x_1, \cdots, x_n)$ given that there are n points in Λ.

Definition 6.1.4 *The density distributions or Janossy densities of a random process μ are the measurable functions j_Λ^n such that:*

$$\begin{aligned} j_\Lambda^n(x_1,\ldots,x_n) &= n!\mathbb{P}(\xi(\Lambda)=n)\pi_{n,\Lambda}(x_1,\ldots,x_n) \text{ for } n\in\mathbb{N}^*, \\ j_\Lambda^0(\emptyset) &= \mathbb{P}(\xi(\Lambda)=0). \end{aligned}$$

Hence the Janossy density $j_\Lambda^n(x_1,\ldots,x_n)$ is the probability that there are exactly n points in Λ located around $x_1,\ldots,x_n$, and no other points anywhere else. For $n=0$, $j_\Lambda^0(\emptyset)$ is the probability that there is no point in Λ. For $n\geq 1$, the Janossy densities satisfy the following properties:

- Symmetry:
$$j_\Lambda^n\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right) = j_{n,\Lambda}\left(x_1,\ldots,x_n\right),$$
for every permutation σ of $\{1,\ldots,n\}$.

- Normalization constraint. For each compact Λ:
$$\sum_{n=0}^{+\infty}\int_{\Lambda^n}\frac{1}{n!}j_\Lambda^n\left(x_1,\ldots,x_n\right)\ \mathrm{d}\lambda(x_1)\ldots\ \mathrm{d}\lambda(x_n)=1.$$

It is clear that the ρ_n's, j_n's, μ should satisfy some relationships. We will not dwell on that here (see the references cited above), we just mention the relation between μ and j_Λ^n, which is:

$$\int_\chi f\left(\xi\right)\ \mathrm{d}\mu(\xi) = \\ \sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n} f(x_1,\ \cdots,\ x_n)\ j_\Lambda^n\left(x_1,\ \cdots,\ x_n\right)\ \mathrm{d}\lambda(x_1)\ldots\ \mathrm{d}\lambda(x_n). \quad (6.1.2)$$

Remark 6.1.2 *If the total number of points in the process is N a.s., then for any $n\geq N+1$ and any compact $\Lambda\subset E$, $j_\Lambda^n\equiv 0$.*

Definition 6.1.5 *For any $n\geq 1$, consider $A_1,\ldots,A_n$ disjoint sets of Λ. In [17], moment measures are defined on $A_1\times\ldots\times A_n$, by:*

$$M_n(A_1\times\ldots\times A_n)=\mathbb{E}[\xi(A_1)\ldots\xi(A_n)].$$

Then M_n is called the n-th moment measure of ξ.

We assume also that these moment measures are alsolutely continuous with respect to the measure λ and define the associated moment densities $m_n(x_1, \ldots, x_n)$. For practical calculation of Janossy densities we use the following relation linking Janossy densities and moment densities. For any $n \geq 1$ and any $x_1, \ldots, x_n \in \Lambda$:

$$j_\Lambda^n(x_1, \ldots, x_n) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \int_{\Lambda^n} m_{n+k}(x_1, \ldots, x_n, y_1, \ldots, y_k) d\lambda(y_1) d\lambda(y_k). \tag{6.1.3}$$

We give an example of Janossy densities.

Proposition 6.1.1 *In the case of a Poisson process with intensity measure σ, the Janossy density is for any $n \geq 1$:*

$$j_\Lambda^n(x_1, \ldots, x_n) = \sigma(x_1) \ldots \sigma(x_n) e^{-\sigma(\Lambda)},$$

and $j_\Lambda^0(\emptyset) = e^{-\sigma(\Lambda)}$.

Proof. For any $n \geq 1$ and $A_1, \ldots, A_n$ disjoint subsets of Λ, the number of points in each subset are independant random variables, and we have:

$$\mathbb{E}\left[\int_{A_1 \times \ldots \times A_n} m_n(x_1, \ldots, x_n) d\lambda(x_1) \ldots d\lambda(x_n)\right] = \mathbb{E}[\xi(A_1) \times \ldots \times \xi(A_n)]$$
$$= \prod_{i=1}^{n} \mathbb{E}[\xi(A_i)] = \sigma(A_1) \ldots \sigma(A_n) = \int_{A_1 \times \ldots \times A_n} \sigma(x_1) \ldots \sigma(x_n) d\lambda(x_1) \ldots d\lambda(x_n).$$

Hence for any $x_1, \ldots, x_n \in \Lambda$:

$$m_n(x_1, \ldots, x_n) = \sigma(x_1) \ldots \sigma(x_n).$$

For any $n \geq 1$ and any $x_1, \ldots, x_n \in \Lambda$ the Janossy densities are given by:

$$\begin{aligned} & j_\Lambda^n(x_1, \ldots, x_n) \\ & = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \int_{\Lambda^k} \sigma(x_1) \ldots \sigma(x_n) \sigma(y_1) \ldots \sigma(y_k) d\lambda(x_1) \ldots d\lambda(x_n) \\ & = \sigma(x_1) \ldots \sigma(x_n) e^{-\sigma(\Lambda)}. \end{aligned}$$

And $j_\Lambda^0 = \mathbb{P}(\xi(\Lambda) = 0) = e^{-\sigma(\Lambda)}$. We can check that:

$$\sum_{n=0}^{+\infty} \frac{1}{n!} \sigma(\Lambda)^n e^{-\sigma(\Lambda)} = 1.$$

■

Then, using Definition 6.1.4, we get the expression of the conditional joint density for Poisson point processes:

$$\pi_{n,\Lambda}(x_1, \ldots, x_n) = \frac{\prod_i \lambda(x_i)}{\lambda(\Lambda)^n}.$$

Indeed, for a Poisson process, conditional on the total number of points in a bounded region, the individual points are independant and identically distributed. This is no longer the case for more general point processes, or for determinantal processes.

We give here the following definitions from [22] concerning the reduced Campbell measure of a point process μ as well as its Papangelou intensity; which describes the local dependence of particles.

Definition 6.1.6 *The reduced Campbell measure of a point process μ is the measure C_μ on the product space $(E \times \xi, \mathcal{B} \otimes \mathcal{F})$ defined by:*

$$C_\mu(A) = \int d\mu(\xi) \sum_{x\in\xi} \mathbf{1}_A(x, \xi \setminus x)$$

where $A \subset \mathcal{B} \times \mathcal{F}$ and $\xi\ x = \xi\ \{x\}$.

Definition 6.1.7 *The reduced compound Campbell measure of a point process μ is the measure $\hat{C}_\mu$ on the product space $(E \times \xi, \mathcal{B} \otimes \mathcal{F})$ defined by:*

$$\hat{C}_\mu(B) = \int d\mu(\xi) \sum_{\alpha\in\chi_0 \alpha\subset\xi} \mathbf{1}_B(\alpha, \xi \setminus \alpha)$$

where $B \subset \mathcal{F}_0 \times \mathcal{F}$.

We give here more definitions from [22]. We consider a measure λ on E and a point process μ such that $C_\mu \ll \lambda \otimes \mu$.

Definition 6.1.8 *Any Radon Nikodym density c of C_μ relative to $\lambda \otimes \mu$ is called the Papangelou conditional intensity of μ.*

The conditional intensity in the sense of Papangelou is a function $c(x, \xi)$ of points $x \in E$ and a configuration ξ. For a measure λ on E, $c(x, \xi)d\lambda(x)$ is the conditional probability of having a particle in dx when the configuraion ξ is given.

Definition 6.1.9 *We also introduce $\hat{c}$, the compound Papangelou intensity, related to the conditional Papangelou intensity by:*

$$\hat{c}\,(\eta,\xi) = c(x_1,\xi)\prod_{i=2}^{n} c\,(x_i, \{x_1, ..., x_{i-1}\} \cup \xi) \ \text{ where } \eta = \{x_1, ..., x_n\}\,.$$

If $\eta = \{x\}$, then $\hat{c}\,(\eta,\xi) = c(\eta,\xi)$

Another important family of processes are Cox processes.

Definition 6.1.10 *A Cox process is a Poisson process with a random intensity σ on the space of Radon measures on E. Its distribution μ_σ satisfies therefore:*

$$\int e^{-\int f d\xi} d\mu_\sigma(\xi) = \mathbb{E}\left[exp\left(-\int_E (1-e^{-f(x)})d\sigma(x)\right)\right],$$

for every positive function f with compact support.

This means that for a Borel subset A of $\mathbb{E}$, we have :

$$\mathbb{P}(\xi(A) = n|\sigma) = \frac{\exp^{-\sigma(A)}}{n!}\sigma(A)^n$$

Proposition 6.1.2 *The Janossy densities of a Cox process with random intensity σ are equal to :*

$$j^n_\Lambda(x_1, ..., x_n) = \mathbb{E}\left[\sigma(x_1)...\sigma(x_n)e^{-\sigma(\Lambda)}\right],$$

for all $n \geq 1$ and $j^0_\Lambda(\emptyset) = \mathbb{E}\left[e^{-\sigma(\Lambda)}\right]$.

Proof. For any subset $A \subset \Lambda$, we have:

$$\mathbb{E}[\xi(A)] = \mathbb{E}[\mathbb{E}[\xi(A)|\sigma]] = \mathbb{E}[\sigma(A)].$$

For $A_1, \ldots, A_n$ disjoint subsets of Λ, the random variables giving the number of points in A_i are independant conditionnally to σ. Hence:

$$\mathbb{E}[\xi(A_1)\ldots\xi(A_n)] = \mathbb{E}[\sigma(A_1)\ldots\sigma(A_n)]$$
$$= \int_{A_1\times\ldots\times A_n} \mathbb{E}\left[\sigma(x_1)\ldots\sigma(x_n)\right] d\lambda(x_1)\ldots d\lambda(x_n).$$

Then the calculation of the factorial moments gives:

$$m_n(x_1, \ldots, x_n) = \mathbb{E}[\sigma(x_1) \ldots \sigma(x_n)].$$

Then for any $n \geq 1$, and any $x_1, \ldots, x_n \in \Lambda$:

$$\begin{aligned}
&j_\Lambda^n(x_1, \ldots, x_n) \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \int_{\Lambda^k} \mathbb{E}[\sigma(x_1) \ldots \sigma(x_n)\sigma(y_1) \ldots \sigma(y_k) d\lambda(y_1) \ldots d\lambda(y_k)] \\
&= \mathbb{E}[\sigma(x_1) \ldots \sigma(x_n) e^{-\sigma(\Lambda)}].
\end{aligned}$$

And $j_\Lambda^0(\emptyset) = \mathbb{P}(\xi(\Lambda) = 0) = \mathbb{E}[e^{-\sigma(\Lambda)}]$. ■

6.2 Fredholm determinants and α-Determinantal point processes

For details on this part, we refer to [20, 38]. For any compact $\Lambda \subset E$, we denote by $L^2(\Lambda, \lambda)$ the set of functions square integrable with respect to the restriction of the measure λ to the set Λ. This becomes a Hilbert space when equipped with the usual norm:

$$\|f\|^2_{L^2(\lambda,\Lambda)} = \int_\Lambda |f(x)|^2 \, \mathrm{d}\lambda(x).$$

For Λ a compact subset of E, P_Λ is the projection from $L^2(E)$ onto $L^2(\Lambda)$, i.e., $P_\Lambda f = f\mathbf{1}_\Lambda$. The operators we will deal with are special cases of the general category of continuous maps from $L^2(E, \lambda)$ into itself.

Definition 6.2.1 *A map T from $L^2(E)$ into itself is said to be an integral operator whenever there exists a measurable function, we still denote by T, such that*

$$Tf(x) = \int_E T(x, y) f(y) \; d\lambda(y).$$

The function T is called the kernel of T.

Definition 6.2.2 *Let T be a bounded map from $L^2(E, \lambda)$ into itself. The map T is said to be trace-class whenever for one complete orthonormal basis (CONB for short) $(h_n, n \geq 1)$ of $L^2(E, \lambda)$,*

$$\sum_{n \geq 1} |(Th_n, h_n)_{L^2}| \text{ is finite.}$$

Then, the trace of T is defined by

$$\operatorname{trace}(T) = \sum_{n\geq 1} (Th_n,\, h_n)_{L^2}.$$

It is easily shown that the notion of trace does not depend on the choice of the CONB. Note that if T is trace-class then T^n also is trace-class for any $n \geq 2$.

Definition 6.2.3 *Let T be a trace class operator. The Fredholm determinant of $(\mathrm{I} + T)$ is defined by:*

$$\operatorname{Det}(\mathrm{I} + T) = exp\left(\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \operatorname{trace}(T^n)\right),$$

where I *stands for the identity operator.*

The practical computations of fractional power of Fredholm determinants involve the so-called α-determinants, which we introduce now.

Definition 6.2.4 *For a square matrix $A = (a_{ij})_{i,j=1\ldots n}$ of size $n \times n$, the α-determinant $det_\alpha\, A$ is defined by:*

$$det_\alpha\, A = \sum_{\sigma\in\Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation is taken over the symmetric group Σ_n, the set of all permutations of $\{1, 2, \cdots, n\}$ and $\nu(\sigma)$ is the number of cycles in the permutation σ.

This is actually a generalization of the well-known determinant of a matrix. Indeed, when $\alpha = -1$, $\det_{-1} A$ is the usual determinant $\det A$. When $\alpha = 1$, $\det_1 A$ is the so-called permanent of A and for $\alpha = 0$, $\det_0 A = \prod_i a_{ii}$. In some cases we can compute explicitly the alpha-determinant. Indeed if A is a n by n matrix where all elements are equal to 1, then :

$$\det_\alpha A = \prod_{j=0}^{n-1} (1 + j\alpha)$$

In the case of non-negative definite matrices of rank one, the α-determinant is equal to some non-negative constant times $\prod_{j=0}^{n-1} (1 + j\alpha)$.

We can then state the following useful theorem (see [37]):

Theorem 6.2.1 *For a trace class integral operator T, if $\| \alpha T \| < 1$, we have:*

$$\mathrm{Det}(\mathrm{I} - \alpha T)^{-\frac{1}{\alpha}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} det_\alpha \left(T(x_i,\, x_j)\right)_{1\le i,j\le n}\ d\lambda(x_1)\dots\ d\lambda(x_n).$$

If $\alpha \in \{-1/m; m \in \mathbb{N}\}$, this is true without the condition $\| \alpha T \| < 1$.

We know from [22] that K can be choosen such that $K(x,x) \ge 0$ for any $x \in E$ and

$$\mathrm{trace}(P_\Lambda K P_\Lambda) = \int_\Lambda K(x,\, x)\ \mathrm{d}\lambda(x),$$

for any compact $\Lambda \subset E$.

We also need to introduce another operator. For a real number α and a compact subset $\Lambda \subset E$, the map $J_{\Lambda,\alpha}$ is formally defined by:

$$J_{\Lambda,\alpha} = (\mathrm{I} + \alpha K_\Lambda)^{-1} K_\Lambda, \tag{6.2.1}$$

so that we have:

$$(\mathrm{I} + \alpha K_\Lambda)\,(\mathrm{I} - \alpha J_{\Lambda,\alpha}) = \mathrm{I}\,.$$

For any compact Λ, the operator $J_{\Lambda,\alpha}$ is also a trace class operator.
In the following theorem ([37]), α-determinantal point processes are formulated in terms of their Laplace transforms, Janossy densities and correlation functions.

Theorem 6.2.2 *Let λ be a Radon measure on E and K a map satisfying hypothesis i, ii and iii. Let $\alpha \in \{2/m; m \in \mathbb{N}\} \cup \{-1/m; m \in \mathbb{N}\}$. Given a nonnegative bounded measurable function f on E with compact support, there exists a unique probability measure $\mu_{\alpha,K_\Lambda,\lambda}$ on the configuration space χ such that:*

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\ d\xi}\right] = \int_\chi e^{-\int f\ d\xi}\ d\mu_{\alpha,K_\Lambda,\lambda}(\xi) = \mathrm{Det}\left(\mathrm{I} + \alpha K_\Lambda[1 - e^{-f}]\right)^{-\frac{1}{\alpha}},$$

where $K_\Lambda[1 - e^{-f}]$ is a bounded operator on $L^2(\Lambda)$ with kernel :

$$K_\Lambda[1 - e^{-f}](x,y) = \sqrt{1 - exp(-f(x))} K_\Lambda(x,y) \sqrt{1 - exp(-f(y))}.$$

This means that for any integer n and any $x_1, \dots, x_n \in \mathbb{N}$ the correlation functions of $\mu_{\alpha,K_\Lambda,\lambda}$ are given, for any $n \ge 1$, by:

$$\rho_{n,\alpha,K_\Lambda}(x_1, \dots, x_n) = det_\alpha \left(K_\Lambda\left(x_i, x_j\right)\right)_{1\le i,j\le n},$$

and for $n = 0$, $\rho_{0,\alpha,K_\Lambda}(\emptyset) = 1$.
For any compact $\Lambda \subset E$, the operator $J_{\Lambda,\alpha}$ is an Hilbert-Schmidt, trace class operator, whose spectrum is included in $[0, +\infty[$. For any $n \in \mathbb{N}^*$, any compact $\Lambda \subset E$, and any $(x_1, \ldots, x_n) \in \Lambda^n$ the Janossy densities are given by:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \ldots, x_n) = \mathrm{Det}\,(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha}\, det_\alpha\,(J_{\Lambda,\alpha}(x_i,\, x_j))_{1 \le i,j \le n}\,.$$

For $n = 0$, we have $j^0_{\Lambda,\alpha,K_\Lambda}(\emptyset) = \mathrm{Det}\,(I + \alpha K_\Lambda)^{-1/\alpha}$.

For $\alpha = -1$, such a process is called a determinantal process since we have, for any $n \ge 1$:

$$\rho_{n,-1,K_\Lambda}(x_1, \ldots, x_n) = \det(K_\Lambda(x_i, x_j))_{1 \le i,j \le n}.$$

For $\alpha = 1$, such a process is called a permanantal process, since we have, for any $n \ge 1$:

$$\rho_{n,1,K_\Lambda}(x_1, \ldots, x_n) = \sum_{\pi \epsilon \Sigma} \prod_{i=1}^{n} K_\Lambda(x_i, x_{\pi(i)}) = \mathrm{per}\,(K_\Lambda(x_i, x_j))_{1 \le i,j \le n}.$$

Remark 6.2.1 *In any dimension, one can ask for necessary and sufficient conditions for the existence of DPP (determinantal point processes) for other values of α. Up to now, existence is granted only for value of α of the form $2/m$ or $-1/m$, where m is an integer (that is the values for which the previous theorem holds). But there is some conjecture that determinantal point processes should exist for any value of α such that $0 \le \alpha \le 2$.*

Remark 6.2.2 *Deriving the Laplace transform, one gets:* $\mathbb{E}[\xi(\Lambda)] = \mathrm{trace}(K_\Lambda)$. *Hence the hypothesis of a locally trace kernel K, thus a trace class kernel K_Λ guarantees the fact that the number of points of the process in any compact Λ is finite.*

Remark 6.2.3 *Using the fact that $j^0_{\Lambda,\alpha,K_\Lambda}(\emptyset) = \mathbb{P}(\xi(\Lambda) = 0)$, another way to write the Janossy densities is:*

$$j^n_{\Lambda,\alpha,K_\Lambda} = \mathbb{P}(\xi(\Lambda) = 0)\, det_\alpha(J_{\Lambda,\alpha}(x_i, x_j))_{1 \le i,j \le n}.$$

For simplicity, in the following, for any $n \in \mathbb{N}^*$, and for any $x_1, \ldots, x_n \in E$, we call $J_{\Lambda,\alpha}$ the n by n matrix, the (i,j)-th term of which is:

$$J_{\Lambda,\alpha}(x_1, \ldots, x_n)_{i,j} = J_{\Lambda,\alpha}(x_i, x_j).$$

The expression $\det_\alpha J_{\Lambda,\alpha}(x_i, x_j)_{1\le i,j\le n}$ is now denoted $\det_\alpha J_{\Lambda,\alpha}(x_1, \dots, x_n)$. For any finite random configuration $\xi = (x_1, \dots, x_n)$, we call $J_{\Lambda,\alpha}(\xi)$ the matrix with terms $J_{\Lambda,\alpha}(x_i, x_j)$. This notation takes into account the fact that the random configuration has a random number of points. The size of the matrix $J_{\Lambda,\alpha}(\xi)$ is equal to the number of points in the configuration ξ. The relation between $J_{\Lambda,\alpha}(x_1, \dots, x_n)$ and $J_{\Lambda,\alpha}(\xi)$ appears in the following equation. Let ξ be a random configuration characterized by the measure $\mu_{\alpha,K_\Lambda,\lambda}$. For any function $F : \mathbb{R} \to \mathbb{R}^+$:

$$
\begin{aligned}
&\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}[F(\det_\alpha\ J_{\Lambda,\alpha}(\xi))] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\det_\alpha\ J_{\Lambda,\alpha}(x_1, \dots, x_n)) j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \dots, x_n)\ \mathrm{d}\lambda(x_1) \dots\ \mathrm{d}\lambda(x_n).
\end{aligned}
$$

Remark 6.2.4 *A determinantal process with $K_\Lambda(x, y) = \mathbf{1}_{\{x=y\}}$ is a Poisson process with intensity λ. We can check that $J_{\Lambda,\alpha}$ has kernel $J_{\Lambda,\alpha}(x, y) = \mathbf{1}_{\{x=y\}}$.*

Another way to retrieve a Poisson process is to let α go to 0.

Theorem 6.2.3 *When α tends to 0, $\mu_{\alpha,K,\lambda}$ converges narrowly to a Poisson measure of intensity $K(x, x)\ d\lambda(x)$.*

Proof. For any nonnegative f, for any $n \ge 1$,

$$
0 \le \operatorname{trace}\left((K_\Lambda[1 - e^{-f}])^n\right) \le \operatorname{trace}\left(K_\Lambda[1 - e^{-f}]\right),
$$

hence,

$$
\begin{aligned}
\int_\chi \exp\left(-\int f\ \ \mathrm{d}\xi\right)\ \mathrm{d}\mu_{\alpha,K_\Lambda,\lambda}(\xi) &= \operatorname{Det}\left(\mathrm{I} + \alpha K_\Lambda[1 - e^{-f}]\right)^{-1/\alpha} \\
&= \exp\left(-\frac{1}{\alpha}\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \alpha^n \operatorname{trace}((K_\Lambda[1 - e^{-f}])^n)\right) \\
\xrightarrow{\alpha\to 0} \exp\left(-\operatorname{trace}(K_\Lambda(1 - e^{-f}))\right) &= \int_E (1 - e^{-f(x)}) K_\Lambda(x,\ x) d\lambda(x). \quad (6.2.2)
\end{aligned}
$$

Thus, when α goes to 0, the measure $\mu_{\alpha,K_\Lambda,\lambda}$ tends towards a measure that we call $\mu_{0,K_\Lambda,\lambda}$. According to (6.2.2), $\mu_{0,K_\Lambda,\lambda}$ is a Poisson process with intensity $K_\Lambda(x,\ x) d\lambda(x)$. ■

Here we give some widely known examples of determinantal processes and we write down the expression of their correlation kernels.
The sine process on $\mathbb{R}$ corresponds to the sine kernel:

$$K^{sine}(x,y) = \frac{\sin(x-y)}{(x-y)} = \int_{-1/2}^{1/2} e^{2i\pi tx} e^{-2i\pi ty} dt.$$

The Fourier transform of the corresponding integral operator K^{sine} in $L^2(\mathbb{R})$ is the operator of multiplication by an indicator function of an interval; hence K^{sine} is a self-adjoint projection operator.

Another example is the Airy point process on $\mathbb{R}$, defined by the Airy kernel:

$$K^{Airy}(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y} = \int_0^{+\infty} Ai(x+t)Ai(y+t)dt,$$

where $Ai(x)$ stands for the classical Airy function. The integral operator K^{Airy} can be viewed as a spectral projection operator for the differential operator $\frac{d^2}{dx^2} - x$ that has the shifted Airy functions $\{Ai(x+t)\}_{t\in\mathbb{R}}$ as eigenfunctions.

Now let us write down examples of determinantal processes and their associated kernels on the complex plane, and show their link with eigenvalues of certain random matrices.
Let $U(n)$ denote the group of $n \times n$ unitary matrices. The circular unitary ensemble introduced by Dyson [19] is the set of eigenvalues of a random U sampled from the normalized Haar measure on $U(n)$. Then the eigenvalues of U lie on the circle $S^1 = \{e^{i\theta}; \theta \in [0, 2\pi[\}$ and have density:

$$j^n(\theta_1, \ldots, \theta_n) = \frac{1}{n!(2\pi)^n} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

The eigenvalue density can also be seen as the determinantal point process on S^1 with kernel:

$$K_n(e^{i\theta}, e^{i\phi}) = \frac{1}{2\pi} \sum_{k=0}^{n-1} e^{ik\theta - ik\phi}.$$

Here is another example, due to Ginibre [26]. Let A be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries. Then the eigenvalues of A have density:

$$j^n(z_1, \ldots, z_n) = \frac{1}{\pi^n \prod_{k=1}^{n-1} k!} e^{-\sum_{k=1}^n |z_k|^2} \prod_{i<j} |z_i - z_j|^2.$$

Equivalently, one may say that the eigenvalues of A form a determinantal point process with kernel:

$$K_n(z,w) = e^{-\frac{|z|^2}{2}-\frac{|w|^2}{2}} \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}.$$

However, in the following, for our purpose we choose to focus on α-determinantal processes with a trace class kernel K fo the following form:

$$K_\Lambda(x,y) = \sum_{k=1}^{N} \lambda_k \phi_k(x)\bar{\phi}_k(y)$$

where $(\phi_k)_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Lambda)$. The ϕ_k are normalized eigenvectors of K_Λ with eigenvalues λ_k in $[0,1]$.

Hypothesis 6.2.1 *In this chapter the operator K_Λ is assumed to be of finite rank $N < +\infty$.*

As we will see later, the total number of points is then finite and at most equal to N.
Then, using (6.2.1), we get the expression of the kernel of the operator $J_{\alpha,\Lambda}$. For any $x,y\in\Lambda$:

$$J_{\alpha,\Lambda}(x,y) = \sum_{k=1}^{N} \frac{\lambda_k}{1+\alpha\lambda_k}\phi_k(x)\bar{\phi}_k(y).$$

When $\alpha = -1$, the operator J_Λ is such that $K_\Lambda = J_\Lambda(I-J_\Lambda)^{-1}$ and has kernel:

$$J_\Lambda(x,y) = \sum_{k=1}^{N} (\frac{\lambda_k}{1-\lambda_k})\phi_k(x)\bar{\phi}_k(y).$$

One can't help but notice the case where $\alpha = -1$ and the restricted operator K_Λ admits the eigenvalue 1 (for example $\lambda_k = 1$ for at least one value of k). In that case, $\mathrm{Det}(I-K_\Lambda) = 0$ and J_Λ is not defined. However, using Lemma 3.4 of Shirai and Takahashi [37], even in the degenerated case where $\mathrm{Det}(I-K_\Lambda) = 0$ it is still possible to define the Janossy densities and to abuse the notation $\mathrm{Det}(I-K_\Lambda)\mathrm{det}J_\Lambda(x_1,\ldots,x_n)$.
In this Chapter however, we remain in the case where J_Λ is well defined.

Proposition 6.2.1 *When $\alpha = -1$, for any of the kernels J_Λ previously defined, an expression of $\det J(x_1,\ldots,x_n)$, for any $n \geq 1$ and any $x_1,\ldots,x_n \in$*

Λ *is:*

$$detJ_\Lambda(x_1,\dots,x_n)=\left(\prod_k\frac{\lambda_k}{1-\lambda_k}\right)det(\phi_i(x_j))_{1,\leq i,j\leq n}det(\bar{\phi}_i(x_j))_{1,\leq i,j\leq n}. \tag{6.2.3}$$

With the hypothesis that $0\leq\lambda_k<1$, one can check that the product of the eigenvalues in the equation above is nonnegative. Thus one can check that the Janossy densities are positive for a determinantal process.

Proof. With this choice of kernel, for any $n\geq 1$ and any $x_1,\dots,x_n\in\Lambda$, the matrix $J_\Lambda(x_1,\dots,x_n)$ can be written :

$$J_\Lambda(x_1,\dots,x_n)=\begin{pmatrix} J_\Lambda(x_1,x_1) & \dots & J_\Lambda(x_1,x_n)\\ \dots & \dots & \dots\\ J_\Lambda(x_n,x_1) & \dots & J_\Lambda(x_n,x_n)\end{pmatrix}$$

$$=\begin{pmatrix} \sqrt{\frac{\lambda_1}{1-\lambda_1}}\phi_1(x_1) & \dots & \sqrt{\frac{\lambda_1}{1-\lambda_1}}\phi_n(x_1)\\ \dots & \dots & \dots\\ \sqrt{\frac{\lambda_n}{1-\lambda_n}}\phi_1(x_n) & \dots & \sqrt{\frac{\lambda_n}{1-\lambda_n}}\phi_n(x_n)\end{pmatrix}\begin{pmatrix} \sqrt{\frac{\lambda_1}{1-\lambda_1}}\bar{\phi}_1(x_1) & \dots & \sqrt{\frac{\lambda_n}{1-\lambda_n}}\bar{\phi}_1(x_n)\\ \dots & \dots & \dots\\ \sqrt{\frac{\lambda_1}{1-\lambda_1}}\bar{\phi}_n(x_1) & \dots & \sqrt{\frac{\lambda_n}{1-\lambda_n}}\bar{\phi}_n(x_n)\end{pmatrix}.$$

Then, the determinant of this matrix is:

$$\mathrm{det}J_\Lambda(x_1,\dots,x_n)=\left(\prod_k\frac{\lambda_k}{1-\lambda_k}\right)\mathrm{det}(\phi_i(x_j))_{1,\leq i,j\leq n}\mathrm{det}(\bar{\phi}_i(x_j))_{1,\leq i,j\leq n}.$$

Hence the result.

■

For $\alpha=2$, a 2 permanental point process is in fact a Cox process based on a Gaussian random field. We know for sure that there exists X a centered Gaussian random field on E such that:

$$\mathbb{E}^{\mathbb{P}}\left[\int_\Lambda X^2(x)\ \mathrm{d}\lambda(x)\right]=\mathrm{trace}(K_\Lambda), \tag{6.2.4}$$

for any compact $\Lambda\subset E$ and

$$\mathbb{E}^{\mathbb{P}}\left[X(x)X(y)\right]=K(x,\,y)\ \lambda\otimes\lambda\ \text{a.s.}, \tag{6.2.5}$$

where $\mathbb{P}$ is the probability measure on the probability space supporting X. Then the Cox process of random intensity $X^2(x)\ \mathrm{d}\lambda(x)$ has the same distribution as $\mu_{2,K,\lambda}$. Indeed, it follows from the formula:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int(1-e^{-f(x)})X^2(x)\ \mathrm{d}\lambda(x)\right)\right]=\mathrm{Det}(I+2(1-e^{-f})K)^{-1/2}.$$

For $\alpha = 1$, permanental processes can also be interpreted as Cox processes. For this case, we refer to the work of [23]. If F is chosen as the following:

$$F(z) = \sum_{k=1}^{N} \sqrt{\lambda_k} a_k \phi_k(z),$$

where the a_k are independant standard complex Gaussian random variables. The difference with the previous example is the introduction of complex random variables.
Let ξ be a Poisson process with random intensity $|F|^2$, ie a Cox process. Then ξ is a permanental process with kernel $K_\Lambda(x, y) = \mathbb{E}[F(x)\bar{F}(y)]$. We can check that:

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \mathbb{E}[a_k \bar{a}_l \phi_k(x) \bar{\phi}_l(y)] = K_{\Lambda(x,y)}.$$

And also that:

$$\mathbb{E}\left[\int_\Lambda |F(x)|^2\right] = \sum_{k=1}^{N} \lambda_k = Tr(K_\Lambda).$$

The following lemma from [37] shows the relation between determinantal, permanental and α-determinantal processes.

Lemma 6.2.1 *(from [37]). If $-1/\alpha$ is a positive integer, the process χ is a union of $-1/\alpha$ i.i.d. copies of the determinantal process with kernel $-\alpha K_\Lambda$. Similarly, if $1/\alpha$ is a positive integer, the process is a union of $1/\alpha$ i.i.d. copies of the permanental process with kernel αK_Λ.*

In the case of determinantal processes, the conditional Papangelou intensities have a straightforward expression, they are given by ratios of determinants. Indeed, [22] gives that:

Theorem 6.2.4 *From [22]. Let $\xi = \{x_1, \ldots, x_n\}$ be a configuration. Let $\eta = \{y_1, \ldots, y_k\}$ be another configuration such that $\xi \subset \Lambda \backslash \eta$. For each Λ, a version of the conditional Papangelou intensity $\hat{c}_\Lambda$ is given by:*

$$\hat{c}_\Lambda(\eta, \xi) = \frac{detJ_\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_k)}{detJ_\Lambda(x_1, \ldots, x_n)},$$

or 0, when then denominator vanishes.
In Theorem 3.1 of [22], it is also shown that:

$$\hat{c}_\Lambda(\eta, \xi) = \frac{detJ_\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_k)}{detJ_\Lambda(x_1, \ldots, x_n)} \leq detJ_\Lambda(y_1, \ldots, y_k).$$

From now on, throughout this chapter, we consider that $d\lambda(x)$ is the Lebesgue measure, and we denote it dx.

6.3 Distribution of the number of points in a subset of Λ

One of the first steps in order to achieve a probabilistic study of an α-determinantal process ξ is to know the probability distribution of the total number of points in compact Λ. Thus, in this section, we consider the probability $\mathbb{P}(\xi(\Lambda) = n)$, which is to say the distribution of the number of points that fall in Λ. In [23], this quantity is expressed in terms of usual random variables. Then we seek the distribution of the number of points of ξ in any subset $D \subset \Lambda$.

At this point, we mention the following remark from [39].

Remark 6.3.1 *For any determinantal point process, with probability* 1 *no two points coincide.*

First, in order to have general estimations of the number of points in Λ, we recall a theorem which extended version is in [39].

Theorem 6.3.1 *([39]). If $Tr(K_\Lambda)$ is finite, the probability of the event that the number of all points in finite is* 1*.*
If K_Λ is a finite rank operator, the number of points is less or equal to N with probability 1*.*

Here by hypothesis $Tr(K_\Lambda) < +\infty$, then number of points is finite and the second part gives the maximal number of points in Λ. Then, more precisely:

Lemma 6.3.1 *From [37]. For a determinantal process, if* 1 *is a eigenvalue m times, then $\mathbb{P}(\xi(\Lambda \geq m)) = 1$.*

There are also particular cases where the number of points is fixed. We recall the following Lemma from [23]:

Lemma 6.3.2 *For ξ determinantal process on Λ with kernel:*

$$K_\Lambda = \sum_{k=1}^{N} \phi_k(x)\bar{\phi}_k(y),$$

the number of points $\xi(\Lambda)$ in compact Λ is N almost surely.

Proof. We use the fact that for this finite rank operator, $\xi(\Lambda) \leq N$ .a.s. Moreover:

$$\mathbb{E}[\xi(\Lambda)] = \int_\Lambda \rho_1(x)dx = Tr(K_\Lambda) = \int_\Lambda K_\Lambda(x,x)dx = \sum_{k=1}^{N} \int_\Lambda |\phi_k(x)|^2 dx = N,$$

using the fact that the $\phi_i, i \in \mathbb{N}$ are orthonormal. Hence $\xi(\Lambda) = N$ .a.s. ■
This shows why it is practical to use these so-called projection kernels where the number of points is fixed. This is however no longer the case for more general kernels, where we only know that $\xi(\Lambda) \leq N$ a.s.

For our purpose we use the following property from [23]:

Lemma 6.3.3 *The number of points of the process ξ that fall in the compact Λ has the distribution of a sum of independent Bernoulli random variables with parameters (λ_k).*
Suppose ξ is a permanental point process with the same kernel K. The number of points of the process ξ that fall in the compact Λ has the distribution of a sum of independent geometric $(\frac{\lambda_k}{\lambda_k+1})$ random variables.
Suppose ξ is an alpha-determinantal point process with a the same kernel K. The number of points of the process ξ that fall in the compact Λ has the distribution of :

- *a sum of independent Binomial $(-1/\alpha, \alpha\lambda_k)$ random variables, if $-1/\alpha$ is a positive integer.*
- *a sum of independent Negative Binomial $(1/\alpha, \frac{\alpha\lambda_k}{\alpha\lambda_k+1})$ random variables, if $\alpha > 0$.*

The last part of the Lemma can be deduced using the superposition property (Lemma 6.2.1).

Remark 6.3.2 *The number of points in Λ of the process $\mu_{0,K_\Lambda,\lambda}$ is in fact a Poisson random variable with parameter $\sum_{k=1}^{N} \lambda_k$.*

Proof. We know that the α-determinantal process characterized by $\mu_{0,K_\Lambda,\lambda}$ is a Poisson process with intensity $K_\Lambda(x,x)d\lambda(x)$. Hence the total number of points in compact Λ is a Poisson random variable with parameter:

$$\begin{aligned}
\int_\Lambda K_\Lambda(x,x)d\lambda(x) &= \int_\Lambda \sum_{k=1}^{N} \lambda_k \phi_k(x)\bar{\phi}_k(x)d\lambda(x) \\
&= \sum_{k=1}^{N} \lambda_k \int_\Lambda |\phi_k(x)|^2 d\lambda(x) \\
&= \sum_{k=1}^{N} \lambda_k.
\end{aligned}$$

Hence the result.
■

In the following we call $M(u)$ the Moment Generating function (MGF) of the total number of points $\xi(\Lambda)$ in Λ:

$$M(u) = \mathbb{E}\left[e^{u\xi(\Lambda)}\right].$$

Proposition 6.3.1 *The Moment Generating Function of $\xi(\Lambda)$ for a determinantal process ($\alpha = -1$) is:*

$$M(u) = exp\left(\sum_{k=1}^{N} \log(1 - \lambda_k(1 - e^u)\right).$$

The MGF of $\xi(\Lambda)$ for a permanental process is:

$$M(u) = \prod_{k=1}^{N} \frac{1}{1 + \lambda_k(1 - e^u)}.$$

The MGF of $\xi(\Lambda)$ for an α-determinantal process with $\alpha < 0$ is:

$$M(u) = exp\left(-\frac{1}{\alpha}\sum_{k=1}^{N} \log(1 + \alpha\lambda_k(1 - e^u))\right).$$

The MGF of $\xi(\Lambda)$ for an α-determinantal process with $\alpha > 0$ is: For $\alpha > 0$, the MGF of the total number of points is:

$$M(u) = \left(\prod_{k=1}^{N} \frac{1}{1 + \lambda_k(1 - e^{-u})}\right)^{1/\alpha}.$$

Proof. We recall that N is the rank of the integral operator. For $\alpha = -1$, using Lemma 6.3.3 the total number of points L has the distribution of a sum of N independant random variables $X_k \sim$ Bernoulli(λ_k):

$$\begin{aligned}
M(u) &= \mathbb{E}\left[e^{u\xi(\Lambda)}\right] = \mathbb{E}\left[e^{u\sum_{k=1}^{N} X_k}\right] \\
&= \prod_{k=1}^{N} \mathbb{E}\left[e^{uX_k}\right] = \prod_{k=1}^{N} (e^u\lambda_k + 1 - \lambda_k) \\
&= \exp\left(\sum_{k=1}^{N} \log(1 - \lambda_k(1 - e^u)\right).
\end{aligned}$$

More generally, for $\alpha < 0$, using Lemma 6.3.3 the number of points in Λ has the same distribution as the sum of N independant random variables X_k such that: $X_k \sim \text{Binomial}(-1/\alpha, -\alpha\lambda_k)$. We remind that $-1/\alpha$ is an integer. Then the MGF of the total number of points is:

$$
\begin{aligned}
&M(u) = \mathbb{E}\left[e^{u\xi(\Lambda)}\right] = \prod_{k=1}^{N} \mathbb{E}\left[e^{uX_k}\right] \\
&= \prod_{k=1}^{N} \sum_{n=0}^{-1/\alpha} \frac{(-1/\alpha)!}{n!(-1/\alpha - n)!} e^{un} (-\alpha\lambda_k)^n (1 + \alpha\lambda_k)^{-1/\alpha - n} \\
&= \prod_{k=1}^{N} (1 + \alpha\lambda_k(1 - e^u))^{-1/\alpha} \\
&= \exp\left(-\frac{1}{\alpha} \sum_{k=1}^{N} \log(1 + \alpha\lambda_k(1 - e^u))\right).
\end{aligned}
$$

When $\alpha \to 0$, it appears that:

$$
\begin{aligned}
M(u) &= \exp\left(-\frac{1}{\alpha} \sum_{k=1}^{N} \log(1 + \alpha\lambda_k(1 - e^u))\right) \\
&\to_{\alpha \to 0} \exp\left(-\left(\sum_{k=1}^{N} \lambda_k\right)(1 - e^u)\right),
\end{aligned}
$$

which is exactly the MGF of a Poisson random variable with parameter $\sum_{k=1}^{N} \lambda_k$.

Similarly, we consider the case where $\alpha > 0$. For $\alpha = 1$, the number of points has the same law as the sum of N independant variables $X_k \sim \text{Geom}(\frac{\lambda_k}{\lambda_k + 1})$. That is to say:

$$
\mathbb{P}(X_k = n) = \left(\frac{\lambda_k}{\lambda_k + 1}\right)^n \frac{1}{\lambda_k + 1},
$$

for any $n \geq 0$. Then the Laplace transform:

$$M(u) = \mathbb{E}\left[e^{u\xi(\Lambda)}\right] = \prod_{k=1}^{N} \mathbb{E}\left[e^{uX_k}\right]$$
$$= \prod_{k=1}^{N} \sum_{n=0}^{+\infty} e^{un} \left(\frac{\lambda_k}{\lambda_k+1}\right)^n \frac{1}{\lambda_k+1}$$
$$= \prod_{k=1}^{N} \frac{1}{1+\lambda_k(1-e^u)}.$$

For $\alpha > 0$, using the superposition property or by direct calculations with the distributions of negative binomial variables, whe show that the MGF of the total number of points is:

$$M(u) = \left(\prod_{k=1}^{N} \frac{1}{1+\alpha\lambda_k(1-e^u)}\right)^{1/\alpha}.$$

And when $\alpha \to 0$, it appears that:

$$\begin{aligned} M(u) &= \left(\prod_{k=1}^{N} \frac{1}{1+\alpha\lambda_k(1-e^u)}\right)^{1/\alpha} \\ &= \exp\left(-\frac{1}{\alpha}\sum_{k=1}^{N} \log(1+\alpha\lambda_k(1-e^u))\right) \\ &\to_{\alpha\to 0} \exp\left(\sum_{k=1}^{N} \lambda_k(1-e^u)\right). \end{aligned}$$

■

Remark 6.3.3 *For any α, we have:*

$$\mathbb{E}[\xi(\Lambda)] = \sum_{k=1}^{N} \lambda_k.$$

As we have seen, the distribution of the total number of points in Λ depends only on the eigenvalues λ_k. Now we examine what this becomes on

on a subset $D \subset \Lambda$.
Let $D \subset \Lambda \subset \mathbb{R}^d$ and an operator K_Λ defined as before. The operator K_D is the restriction of K_Λ to D. Its kernel $K_D(x,y)$ can be written:

$$K_D(x,y) = \sum_{k=1}^{+\infty} \lambda_k^D \psi_k(x) \bar{\psi}_k(y),$$

where $(\psi_k, k \in \mathbb{N})$ is an orthonormal basis of $L^2(D)$, and $(\lambda_k^D)_{k=1,\dots,N}$ are the eigenvalues of the operator K_D, see [23]. In this sum only N terms at most are different from 0. The properties concerning the number of points of the process in a compact still hold. But because for the moment, only the eigenvalues λ_k are known, it is first necessary to determine the $(\lambda_k^D)_{k=1,\dots,N}$ as functions of the λ_k. This is the purpose of this section.

Remark 6.3.4 *The eigenvalues of the operator K_Λ restricted to $D \subset \Lambda$ verify :*

$$\sum_k \lambda_k^D < \infty.$$

The kernel of the operator K_D can also be expressed as:

$$K_D(x,y) = \mathbf{1}_D(x) K_\Lambda(x,y) \mathbf{1}_D(y),$$

which can also be written:

$$K_D(x,y) = \sum_{k=1}^{N} \lambda_k \phi_k^D(x) \bar{\phi}_k^D(y).$$

which is an operator from $L^2(D)$ into $L^2(D)$, with same rank N as K_Λ. The family $(\phi_k^D, k \in \mathbb{N})$ is the restriction of $(\phi_k, k \in \mathbb{N})$ to D and is not an orthonormal basis on $L^2(D)$. Then we have the following theorem:

Theorem 6.3.2 *With this set of hypothesis, it is possible to compute the eigenvalues of the operator K_D, for any $l \geq 1$:*

$$\lambda_l^D = \sum_{i=1}^{N} \lambda_i \left(\int_D \phi_i^D(x) \bar{\psi}_l(x) d\lambda(x) \right) \left(\int_D \bar{\phi}_i^D(y) \psi_l(y) d\lambda(y) \right).$$

Proof. For any function f:

$$K_D f(x) = \int_D K_D(x,y) f(y) d\lambda(y) = \sum_{k=1}^{+\infty} \lambda_k^D \psi_k(x) \int_D f(y) \bar{\psi}_k(y) d\lambda(y).$$

In particular,

$$\begin{aligned} K_D\psi_l(x) &= \sum_{k=1}^{+\infty} \lambda_k^D \psi_k(x) \int_D \psi_l(y)\bar{\psi}_k(y)d\lambda(y) \\ &= \lambda_l^D \psi_l(x). \end{aligned}$$

Then the scalar product of the vectors $K_D(x,y)\psi_l(x)$ and $\bar{\psi}_l(x)$ in $L^2(E,\lambda)$ is:

$$\langle K_D(x,y)\psi_l(x), \bar{\psi}_l(x)\rangle = \lambda_l^D \int_D \psi_l(x)\bar{\psi}_l(x)d\lambda(x) = \lambda_l^D.$$

On the other hand:

$$K_D f(x) = \sum_{k=1}^{N} \lambda_k \phi_k^D(x) \int_D \bar{\phi}_k^D(y) f(y) d\lambda(y).$$

In particular:

$$K_D\psi_l(x) = \sum_{k=1}^{N} \lambda_k \phi_k^D(x) \left(\int_D \bar{\phi}_k^D(y)\psi_l(y)d\lambda(y) \right).$$

And:

$$\langle K_D\psi_l(x), \bar{\psi}_l(x)\rangle = \sum_{k=1}^{N} \lambda_k \left(\int_D \phi_k^D(x)\bar{\psi}_l(x)d\lambda(x) \right) \left(\int_D \bar{\phi}_k^D(y)\psi_l(y)d\lambda(y) \right).$$

The proof is thus complete. ■ When it is well-defined, it is then possible to write the expression of the kernel of the corresponding operator $J_{\alpha,D}$ as:

$$J_{\alpha,D}(x,y) = \sum_{k=1}^{N} \lambda_k^D \psi_k(x)\bar{\psi}_k(y),$$

where the eigenvectors are the same as in K_D. Hence, from the expression of the kernel of K_Λ we can deduce the expression of $J_{\alpha,\Lambda}$, with same rank, same eigenvectors ϕ_k but different eigenvalues. Then, from the previous theorem we deduce the relation between the eigenvalues of K_D and K_Λ. Then, using the eigenvalues of K_D we deduce the eigenvalues of $J_{\alpha,D}$.

Here we examine a particular example in the one-dimensional case. Let $0 < t < T$. Assume $K_{[0,T]}$ is an operator from $L^2([0,T])$ into $L^2([0,T])$ with rank N and eigenvalues $(\lambda_k, k = 1,\dots,N)$. In this particual case, we choose $(\phi_k, k = 1,\dots) = (\cos(\frac{2\pi}{T}kx), k = 1,\dots)$ as an orthonormal basis of

$L^2([0,T])$, with normalization $\sqrt{2/T}$. Its kernel $K_{[0,T]}(x,y)$ is then written, for any $x, y \in [0,T]$:

$$K_{[0,T]}(x,y) = \frac{2}{T}\sum_{k=1}^{N} \lambda_k \cos\left(\frac{2\pi}{T}kx\right)\cos\left(\frac{2\pi}{T}ky\right).$$

We consider its restriction $K_{[0,t]}$ to $[0,t]$. The advantage of this example is that we know explicitly the expression of an orthonormal basis on $L^2([0,t])$. We choose $(\psi_k, k = 1, \ldots) = (\cos(\frac{2\pi}{t}kx), k = 1, \ldots)$. The restriction $K_{[0,t]}$ has kernel:

$$K_{[0,t]}(x,y) = \sum_{k=1}^{+\infty} \alpha_k \cos\left(\frac{2\pi}{t}kx\right)\cos\left(\frac{2\pi}{t}ky\right),$$

and the eigenvalues α_k remain to be expressed. Then, using Theorem (6.3.2), we have for any $l \geq 1$, the expression of the eigenvalues α_l of the restriction operator $K_{[0,t]}$:

$$\lambda_l^{[0,t]} = \sum_{k=1}^{N} \lambda_k \left(\int_0^t \cos\left(\frac{2\pi}{T}kx\right)\cos\left(\frac{2\pi}{t}lx\right)dx\right)\left(\int_0^t \cos\left(\frac{2\pi}{T}ky\right)\cos\left(\frac{2\pi}{t}ly\right)dy\right).$$

First example: $N = 3$ and the eigenvalues on $\Lambda = [0,T] = [0,2]$ are $\lambda_1^{[0,2]} = 1$, $\lambda_2^{[0,2]} = 1/2$ and $\lambda_3^{[0,2]} = 1/6$. We observe the values of $\lambda_k^{[0,t]}$ on $t \in [0,2]$ (see Figure 6.3).

When the observation interval $[0,t]$ reduces, the number of points observed decreases. When $t \to 0$, $\lim_{t\to 0} \lambda_l^{[0,t]} \to 0$, for all $l \in 1, \ldots, N$. This means that the total number of points in $[0,t]$ also tends to 0, using Lemma 6.3.3. This is what was expected. Indeed, when the observation interval $[0,t]$ becomes smaller, the number of points of the process observed decreases.

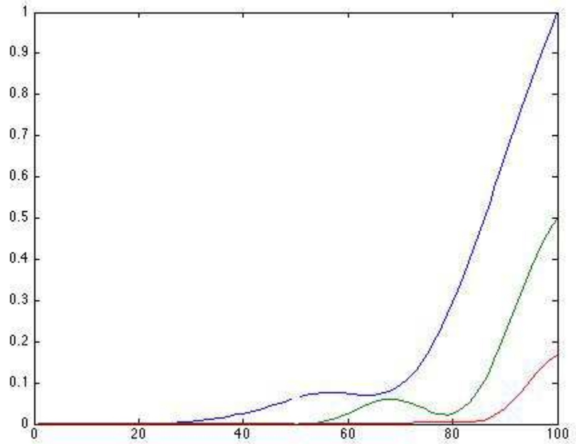


Figure 6.1: *Eigenvalues* $\lambda_k^{[0,t]}$

6.4 Distribution of $\xi_B|\xi_A$

Given two subsets A and B of Λ such that $A \subset B$. We aim to characterize the distribution of a determinantal process ξ_B on B conditionnally to the distribution of ξ_A on A.

Proposition 6.4.1 *Consider a determinantal process ξ, with parameters $\alpha = -1$, an operator K (and the associated operator J) and a reference measure which is here assumed to be the Lebesgue measure on $E = \mathbb{R}^d$. Let $A \subset B \subset \Lambda$. We denote by ξ_A, ξ_B, J_A, J_B the restrictions of the process and of the operators to each of the subsets. Then the distribution of of ξ_B conditionnally to ξ_A has density given by:*

$$\frac{detJ_B(\xi_B)}{detJ_A(\xi_A)detJ_{B\backslash A}(\xi_{B\backslash A})}.$$

Proof. Consider two functions f and g from χ into $\mathbb{R}$. We compute the conditional expectation of ξ_B conditionnally to ξ_A.

$$\begin{aligned}
&\mathbb{E}[f(\xi_A)g(\xi_B)] \\
&= \sum_n \frac{1}{n!} \int_{B^n} g(x_1, \dots, x_n) f(\{x_1, \dots, x_n\} \cap A) \det J_B(x_1, \dots, x_n) dx_1 \dots dx_n \\
&= \sum_n \sum_k \frac{1}{k!(n-k)!} \int_{A^k \times (B\backslash A)^{n-k}} g(x_1, \dots, x_n) f(x_1, \dots, x_k) \det J_B(x_1, \dots, x_n) dx_1 \dots dx_n
\end{aligned}$$

$$= \sum_k \frac{1}{k!} \int_{A^k} f(x_1, \ldots, x_k) \sum_j \frac{1}{j!} \int_{(B\setminus A)^{n-k}} g(x_1, \ldots, x_k, y_1, \ldots, y_j)$$
$$\times\, p(x_1, \ldots, x_k, y_1, \ldots, y_j) \det J_{B\setminus A}(y_1, \ldots, y_j) dy_1 \ldots dy_j \det J_A(x_1, \ldots, x_k) dx_1 \ldots dx_k.$$

where:

$$p(x_1, \ldots, x_k, y_1, \ldots, y_j) = \frac{\det J_B(x_1, \ldots, x_k, y_1, \ldots, y_j)}{\det J_A(x_1, \ldots, x_k) \det J_{B\setminus A}(y_1, \ldots, y_j)}.$$

Because for any subset $A \subset B$ and $t, s \in B$, we have $J_B(t,s) = J_B(t,s)\mathbf{1}_A(t)\mathbf{1}_A(s)$, we have:

$$p(x_1, \ldots, x_k, y_1, \ldots, y_j) = \frac{\det J_B(x_1, \ldots, x_k, y_1, \ldots, y_j)}{\det J_B(x_1, \ldots, x_k) \det J_B(y_1, \ldots, y_j)}.$$

■ Along these lines, we refer to Theorem 7.4.3 in Chapter 7.
And finally, using a Theorem 3.1 from [22], for all $x_1, \ldots, x_k \in A$ and $y_1, \ldots, y_j \in B\setminus A$, we have:

$$p(x_1, \ldots, x_k, y_1, \ldots, y_j) \leq 1.$$

This property could be useful for simulations (with an acception/rejection method). This is the purpose of the following section.

6.5 Simulations

In this section we address the simulation of determinantal processes on $E = \mathbb{R}^d$ and give examples in the particular cases where E is the complex plane or $E = \mathbb{R}$.

6.5.1 Simulation of determinantal processes ($\alpha = -1$)

We first recall related work concerning the simulation of determinantal point processes. Then using the characterization of finite point processes in Definition 6.1.2 and the distribution of $\xi(\Lambda)$, we suggest another method based on the conditional distribution of the points.
An algorithm for the simulation of determinantal processes has been given in [23] for determinantal processes with projection kernel:

$$K_\Lambda(x, y) = \sum_{k=1}^{N} \phi_k(x) \bar{\phi}_k(y).$$

But as we see, this algorithm relies explicitely on the spectral decomposition of the kernel. But for more general kernels this decomposition could be unlikely to be known. They have also considered the particular case of radially symmetric kernels in the complex plane of the form:

$$K(z,w) = \sum_k a_k (z\bar{w})^k,$$

where there are properties concerning the distribution of the absolute values of the points (see details in [23]). The particular case of determinantal processes on a discrete set has also been considered by Moller.

In the remaining part of this section, we suggest two approaches for the simulation of determinantal point processes.
The first approach uses conditional Papangelou intensities. Indeed, the conditional Papangelou intensity is known relatively explicitely for determinantal processes and determines the density of the position of an additional point conditionally to an observed part of a configuration. One can then use this knowledge to create a realization of a determinantal by iterative construction. Thus, conditionnally to a configuration of k points, the Papangelou conditional intensity gives us the probability density of the position of the $(k+1)$-th point (Theorem 6.2.4):

$$\hat{c}_\Lambda(x_{k+1}|x_1,\dots,x_n) = \frac{\det J_\Lambda(x_1,\dots,x_k,x_{k+1})}{\det J_\Lambda(x_1,\dots,x_k)},$$

and by Theorem 6.2.4, we have $\hat{c}_\Lambda(x_{k+1}|x_1,\dots,x_n) \leq J_\Lambda(x_{k+1},x_{k+1})$.
Thus, by an acception/rejection method, it is possible to simulate the position of the $(k+1)$-th point, conditionnally to the realization of the k first points, and so on.
Along the same lines, conditionnally to a configuration $\xi_A = (x_1,\dots,x_k)$ on $A \subset \Lambda$, the previous section gives the distribution of the configuration $\xi_B = (x_1,\dots,x_k,y_1,\dots,y_j)$ on a bigger subset $A \subset B \subset \Lambda$, and its density is proportional to:

$$p(x_1,\dots,x_k,y_1,\dots,y_j) = \frac{\det J_B(x_1,\dots,x_k,y_1,\dots,y_j)}{\det J_B(x_1,\dots,x_k)\det J_B(y_1,\dots,y_j)}.$$

An acception/rejection method can be applied, using the fact that $p(x_1,\dots,x_k,y_1,\dots,y_j) \leq 1$. Thus, starting from a configuration on a subset $A \subset \Lambda$, this gives the possibility of simulating a configuration of a determinantal process on the bigger subset $A \subset B \subset \Lambda$.

We propose another approach for simulations, based on the Janossy densities of the distribution $(x_1, \ldots, x_n)$ (equation (6.1.4)) and on Definition 6.1.2.
Step 1: Generation of a random number N according to the distribution $\mathbb{P}(\xi(\Lambda) = n)$ of the number of points in Λ (Lemma 6.3.3).
Step 2: Conditionnally to the fact that $N = n$, simulation of the vector $(x_1, \ldots, x_n)$, according to the joint distribution $\Pi_{n,\Lambda}(x_1, \ldots, x_n)$.

$$\pi_{n,\Lambda}(x_1, \ldots, x_n) = \frac{j^n_{\Lambda,-1,K_\Lambda}(x_1, \ldots, x_n)}{n!\mathbb{P}(\xi(\Lambda) = n)} = \frac{\mathbb{P}(\xi(\Lambda) = 0)}{n!\mathbb{P}(\xi(\Lambda) = n)} \det J_\Lambda(x_1, \ldots, x_n).$$

This last expression is actually equal to a constant (which value does not matter for the algorithm) times $\det J_\Lambda(x_1, \ldots, x_n)$, that is:

$$\pi_{n,\Lambda}(x_1, \ldots, x_n) = C \det J_\Lambda(x_1, \ldots, x_n).$$

And again this step can be computed by an acception/rejection method.
In any case, a key point is the simulation of a probability density involving the term: $\det J_\Lambda(x_1, \ldots, x_n)$.

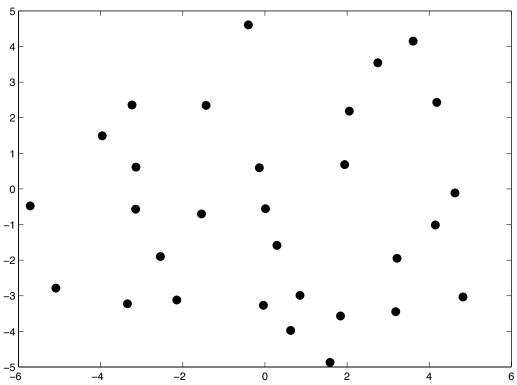


Figure 6.2: Determinantal process ($\alpha = -1$)

Remark 6.5.1 *We remind how this simulation technique would apply to the simple case of a Poisson process, with constant intensity σ, observed on a compact $[0, T] \subset \mathbb{R}$. We recall that the reference measure we consider here is stille the Lebesgue measure on $[0, T]$. The first step is then the simulation of a random Poisson variable N with parameter σT. Then conditionnally to the fact that $N = n$, we simulate independantly n points, each of them according to the uniform distribution on $[0, T]$.*

6.5.2 Simulation of permanental processes

In the case of a permanental process, we use the fact that it is also a Cox process. We use the doubly stochastic construction of a Cox process.
Let us describe an example here. Let K_Λ be a kernel defined as before, with rank N, eigenvectors ϕ_k and eigenvalues λ_k. We generate a vector of size N of independant standard complex gaussian random variables a_k,$k = 1, \dots, N$. Given a vector of size N, we generalte a realization of $|F(z)|^2 = |\sum_{k=1}^{N} \sqrt{\lambda_k} a_k \phi_k(z)|^2$. Conditionnally to this realization, we generate a Poisson process with intensity $|F(z)|^2$. This is a permanental process (with $\alpha = 1$ and kernel K_Λ).

6.5.3 Alpha-determinantal processes

The expressions of Janossy densities and conditional probability distribution previously mentioned still hold for α-determinantal processes. However, they involve α-determinants. While usual determinants are well-known and easily numerically computed, we want to avoid the computation of α-determinants, thus we take advantage of the superposition property.

Using the superposition property (Lemma 6.2.1), we can deduce the simulation of α-determinantal processes from the superposition of independant and identically distributed determinantal or permanental processes. For instance the superposition of two independant and identically distributed determinantal processes with parameters $\alpha = -1$ and $K/2$ (such as the one of Figure 6.5.3) form a $-1/2$-determinantal process with kernel K (Figure 6.5.3).

These figures illustrate how the parameter α controls the strength of the attraction or repulsion between points. Comparing Figures 6.5.1 and 6.5.3 we see that the expectation of the total number of points is the same for both figures but there is less repulsion between the points in Figure 6.5.3, where $\alpha = -1/2$.

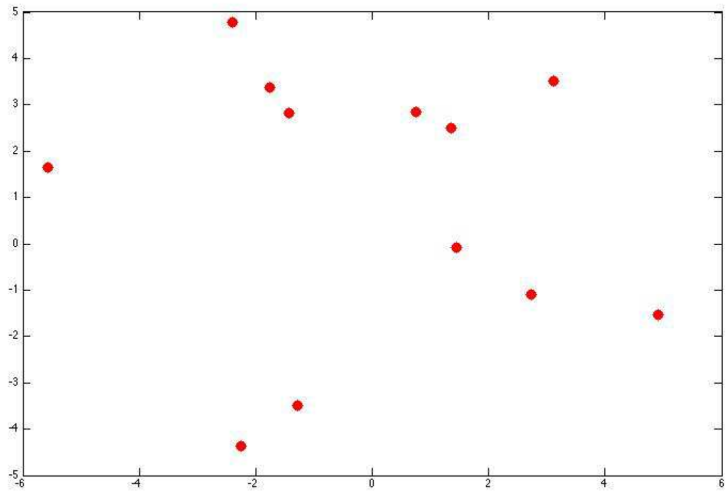


Figure 6.3: Determinantal process with parameters $\alpha = -1$ and $K/2$

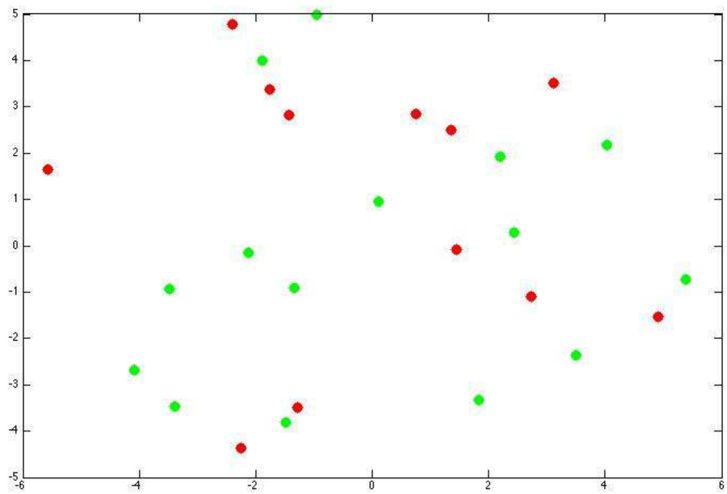
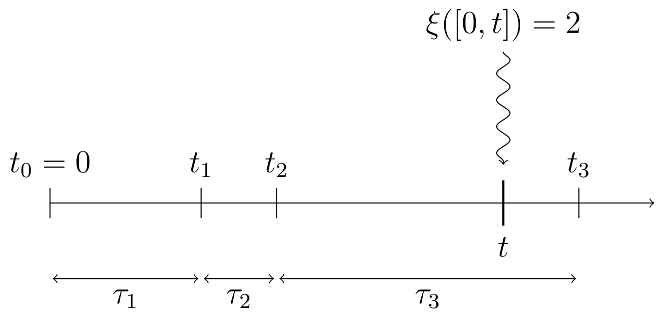


Figure 6.4: α-Determinantal process with parameters $\alpha = -1/2$ and K

6.6 Conditional intensities of determinantal processes

We consider the case of point processes taking values in $E = \mathbb{R}^+$. A realization of a point process is thus a set of dates, which can be ordered.

We take advantage of the order properties of the real line.
For point processes on the real line, it is possible to define a conditional intensity, defined on the intervals $[t_i, t_{i+1}]$, for $i \geq 0$.
Some point processes are easily defined with their conditional intensities, in the sense of Daley and Vere-Jones [17] (for example Hawkes processes). Determinantal point processes however, are more easily defined through their correlation functions or Janossy densities. This section shows that it is possible to go from intensities or Janossy densities or from Janossy densities to intensities.
The essence of this approach is the description of the process through successive conditioning. First we remind the theory of Daley and Vere-Jones [17], then we apply this to determinantal processes.
For simplicity, we suppose observation of the process occurrs over the interval $\Lambda = [0, T] \subset \mathbb{R}$. We denote by $\{t_1, .., t_i, ..t_{N(T)}\}$ the ordered set of points occuring in the interval $[0, T]$. We have : $0 < t_1 < ... < t_i < ....t_{N(T)}$. Here we recall that the random variable $N(T)$ represents the total number of points in the interval $[0, T]$. We denote $t_0 = 0$ and the $\tau_i, i \geq 1$ are the intervals between the occurences. That is: $\tau_i = t_i - t_{i-1}$. They are assumed to be well-defined random variables.



Remark 6.6.1 *An useful property is that the Janossy densities depend on the compact Λ. More precisely, for any $0 \leq t \leq T$, the following indentity holds:*

$$j^0_{[0,t],\alpha,K_\Lambda} = j^0_{[0,T],\alpha,K_\Lambda} + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_t^T \dots \int_t^T j^k_{[0,T],\alpha,K_\Lambda}(u_1, \dots, u_k) du_1 \dots du_k.$$

We give here the definitions of the conditional survivor functions :

$$S_n(u|t_1, ..., t_{n-1}) = \mathbb{P}\left(\tau_n > u|t_1, ..., t_{n-1}\right)$$

The functions $p_n(u|t_1, ..., t_{n-1})$ are the probability densities corresponding to these survivor functions. More precisely:

Lemma 6.6.1 *For a regular point process on $\mathbb{R}^+$ there exists a uniquely determined family of conditional probability density functions and associated survivor functions :*

$$\begin{aligned} S_1(T) &= \mathbb{P}(\tau_1 > T). \\ \forall n \geq 2 \ S_n(t|t_1, ..., t_{n-1}) &= \mathbb{P}\left(\tau_n > t|t_1, ..., t_{n-1}\right) \\ &= 1 - \int_{t_{n-1}}^{t} p_n(u|t_1, ..., t_{n-1})du \end{aligned}$$

defined on $0 < t_1 < \ldots < t_{n-1} < t$. The functions $p_n(t|t_1, ..., t_{n-1})$ have a support carried by the half-line $[t_{n-1}, +\infty]$.

These probability densities can be represented recursively in terms of the Janossy densities, using the following theorem from [17].

Theorem 6.6.1 *The quantity $j^0_{[0,T]}$ is the probability that there are no points between 0 and T. It is equal to the probability $S_1(T)$ that the first point t_1 occurrs only after T.*

$$j^0_{[0,T]} = \mathbb{P}(t_1 > T) = S_1(T). \tag{6.6.1}$$

And for all $n \geq 1$ and all finite intervals $[0, T]$ with $T > 0$:

$$j^n_{[0,T]}(t_1, \ldots, t_n) = p_1(t_1)p_2(t_2|t_1)...p_n(t_n|t_1, ..., t_{n-1})\mathbb{P}(\tau_{n+1} > T|t_1, ..., t_n), \tag{6.6.2}$$

where $0 < t_1 < ... < t_n < T$.
Conversely, given such family of conditional probability densities, for all $t > 0$, equations (6.6.1) and (6.6.2) specify uniquely the distribution of a regular point process on $\mathbb{R}^+$. More precisely, for any $n \geq 1$:

$$\begin{aligned} &p_1(t_1) \ldots p_{n-1}(t_{n-1}|t_1, \ldots, t_{n-2})p_n(t|t_1, ..., t_{n-1}) \\ &= j^n_{[0,T]}(t_1, ..., t_{n-1}, t) + \sum_{k=n}^{+\infty} \frac{1}{(k-n)!} \int_t^T ... \int_t^T j^k_{[0,T]}(t_1, ..., t_{n-1}, t, u_1, .., u_k)du_1...du_k. \end{aligned}$$

Definition 6.6.1 *Instead of specifying the conditional densities p_n, it is possible to use hazard functions instead. The hazard function is defined by :*

$$h_n(t|t_1, ..., t_{n-1}) = \frac{p_n(t|t_1, ..., t_{n-1})}{S_n(t|t_1, ..., t_{n-1})}.$$

Conversely, knowing the probability densities p_n, it is possible to express them in terms of their hazard functions:

$$p_n(t|t_1, ..., t_{n-1}) = h_n(t|t_1, ..., t_{n-1})exp\left(-\int_{t_{n-1}}^{t} h_n(u|t_1, ..., t_{n-1})du\right).$$

Definition 6.6.2 *Given a strictly increasing sequence of points* $0 < t_1 < \ldots < t_{n-1}$*, the conditional intensity function for a regular point process on* $\mathbb{R}^+$ *is the function* λ_t^* *defined piecewise by :*

$$\begin{aligned} \lambda_t^* &= h_1(t) \text{ on } 0 < t \leq t_1. \\ \forall n \geq 2 \ \lambda_t^* &= h_n(t|t_1, ..., t_{n-1}) \text{ on } t_{n-1} < t \leq t_n. \end{aligned}$$

Example 6.6.1 *A class of point processes for which the conditional intensities have a very straightforward expression are Hawkes processes:*

$$\begin{aligned} \lambda_t^* &= c \text{ for } t < t_1 \\ &= c + \delta e^{\kappa(t-T_1)} \text{ for } t_1 \leq t < t_2 \\ &\ldots \\ &= c + \delta \sum_{i=1}^{n-1} e^{\kappa(t-t_i)} \text{ for } t_{n-1} \leq t < t_n. \end{aligned}$$

Theorem 6.6.2 *For an* α*-determinantal process with kernel* $K_{[0,T]}$*, defined as before, we have, on any* $t_{n-1} < t \leq t_n$*, for* $n \geq 2$*, the conditional intensity:*

$$\lambda_t^* = \frac{p_n(t|t_1, ..., t_{n-1})}{1 - \int_{t_{n-1}}^{t} p_n(u|t_1, ..., t_{n-1})} du,$$

where for any $n \geq 1$*,* $p_n(t_n|t_1, ..., t_{n-1})$ *is given by 6.6.3.*

Proof. For determinantal processes:

$$p_1(t) = \mathrm{Det}(I - K_{[0,T]}) \sum_{k=1}^{+\infty} \frac{1}{(k-1)!} \int_t^T ... \int_t^T \det J_{[0,T]}(t, u_1, .., u_k) du_1 ... du_k$$

That is:

$$p_1(t) = j^1_{[0,t],-1,K_{[0,t]}}(t).$$

It is possible to obtain Janossy density when one knows the conditional intensity or conversely.
For $n \leq 2$, $p_2(t|t_1)$ is deduced from the relation:

$$p_1(t_1) p_2(t|t_1) = j^2_{[0,t],-1,K_{[0,t]}}(t_1, t),$$

where j^2 is the two-points Janossy density.
And similarly, for any $n \geq 1$ and for $0 \leq t_1 \leq ... \leq t_n \leq t$:

$$p_n(t_n|t_1, ..., t_{n-1}) \ldots p_2(t_2|t_1) p_1(t_1) = j^n_{[0,t],-1,K_{[0,t]}}(t_1, ..., t). \tag{6.6.3}$$

This gives us all the functions $p_n(t_n|t_1,\ldots,t_n)$. Thus we get:

$$\text{on } t_{n-1} < t \leq t_n \ \forall n \geq 2$$

$$\lambda_t^* = h_n(t|t_1,...,t_{n-1}) = \frac{p_n(t|t_1,...,t_{n-1})}{1 - \int_{t_{n-1}}^{t} p_n(u|t_1,...,t_{n-1})du} \text{ with the previous } p_n$$

Hence the result. ■

Chapter 7

Integration by parts formula for determinantal and permanental point processes

Contents

Ce chapitre a pour objet une formule d'intégration par parties pour les processus déterminantaux et permanentaux. Il s'agit d'un article paru dans *Journal of Functional Analysis* [15]. Dans ce chapitre nous établissons d'abord un résultat de quasi-invariance pour les processus déterminantaux. Nous montrons que si les points sont perturbés le long d'un champ de vecteurs, le processus qui en résulte est toujours un déterminantal, dont la loi est absolument continue par rapport à la distribution d'origine. En se servant de cette formule de quasi-invariance, et en suivant l'approche de Bismut du calcul de Malliavin, nous donnons alors une formule d'intégration par parties pour les processus déterminantaux. Il est possible d'étendre cette formule d'intégration par parties aux processus permanentaux et plus généralement aux processus α-déterminantaux.

Abstract

Determinantal and permanental processes are point processes with a correlation function given by a determinant or a permanent. Their atoms exhibit mutual attraction or repulsion, thus these processes are very far from the uncorrelated situation encountered in Poisson models. We establish a quasi-invariance result : we show that if atoms locations are perturbed along a vector field, the resulting process is still a determinantal (respectively permanental) process, the law of which is absolutely continuous with respect to the original distribution. Based on this formula, following Bismut approach of Malliavin calculus, we then give an integration by parts formula.

7.1 Motivations

Point processes are widely used to model various phenomena, such as arrival times, arrangement of points in space, etc. It is thus necessary to know into details as large a catalog of point processes as possible. The Poisson process is one example which has been widely studied for a long time. Our motivation is to study point processes that generate a more complex correlation structure, such as a repulsion or attraction between points, but still remain simple enough so that their mathematical properties are analytically tractable. Determinantal and permanental point processes hopefully belong to this category. They were introduced in [31] in order to represent configurations of fermions and bosons. Elementary particles belong exclusively to one of these two classes. Fermions are particles like electrons or quarks; they obey the Pauli exclusion principle and hence the Fermi-Dirac statistics. The other sort of particles are particles like photons which obey the Bose-Einstein statistics. The interested reader can find in [42] an illuminating account of the determinantal (respectively permanental) structure of fermions (respectively bosons) ensemble. A mathematical unified presentation of determinantal/permanental point processes (DPPP for short) was for the first time, introduced in [37]. Let χ be the space of locally finite, simple configurations on a Polish space E and K a locally trace-class operator in $L^2(E)$ with a Radon measure λ. For $\alpha \in \mathfrak{A} = \{2/m;\, m \in \mathbb{N}\} \cup \{-1/m,\, , \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers, for any positive, compactly supported f and $\xi = \sum_j \delta_{x_j} \in \chi$, the α-DPPP is the measure, $\mu_{\alpha,\, K,\, \lambda}$, on χ such that

$$\int_\chi e^{-\int f \, \mathrm{d}\xi} \, \mathrm{d}\mu_{\alpha,K,\lambda}(\xi) = \operatorname{Det}\left(\mathrm{I} + \alpha\sqrt{1-e^{-f}}\, K \,\sqrt{1-e^{-f}}\right)^{-\frac{1}{\alpha}}. \tag{7.1.1}$$

The values $\alpha = -1$ and $\alpha = 1$ correspond to determinantal and permanental point processes respectively. Starting from (7.1.1), existence of α-DPPP for any value of α is still a challenge as explained in [39]. Actually, existence is (not easily) proved for $\alpha = \pm 1$ and DPPP for other values of $\alpha \in \mathfrak{A}$ are constructed as superposition of these basic processes. DPPP recently regained interest because they have strong links with the spectral theory of random matrices [27, 39]: for instance, eigenvalues of matrices in the Ginibre ensemble a.s. form a determinantal configuration. DPPP also appear in polynuclear growth [25, 26], non intersecting random walks, spanning trees, zero set of Gaussian analytic functions (see [23] and references there in), etc. Mathematically speaking, a few of their properties are known. The most complete references to date are, to the best of our knowledge, [23, 37] and references there in. The overall impression seems to be that DPPP are rather hard to describe and analyze, their properties being highly dependent of the kernel and its eigenvalues.

Our aim is to investigate further some of the stochastic properties of α-DPPP. In the spirit of [46], we are interested in the differential calculus associated to these processes. We here address the problem within the point of view of Malliavin calculus. To date, Malliavin calculus for point processes has been developed namely for Poisson processes ([3, 6, 7, 10, 18, 35]) and some of their extensions: Gibbs processes [4], marked processes [2], filtered Poisson processes [18], cluster processes [9] and Lévy processes [5, 19]. There exist three approaches to construct a Malliavin calculus framework for point processes: one based on white noise analysis, one based on a difference operator and chaos decomposition and one which relies on quasi-invariance of the law of Poisson process with respect to some perturbations. This is the last track we follow here since neither the white noise framework nor the chaos decomposition exist so far.

We first show that the action of a diffeomorphism of E into itself onto the atoms of an α-DPPP yields another α-DPPP, the law of which is absolutely continuous with the distribution of the original process; a property usually known as quasi-invariance. Then, following the lines of proof of [3, 8, 9]; we can derive an integration by parts formula for the differential gradient as usually constructed on configuration spaces. This gives another proof of the closability of the Dirichlet form canonically associated to an α-DPPP as in [46].

This paper is organized as follows. In part 7.2, we give definitions concerning point processes and α-DPPP. In part 7.3, we prove the quasi-invariance for α-DPPP. Then, in Section 7.4, we compute the integration by parts formula. We begin by determinantal point processes and then extend to α-determinantal point processes. Permanental processes are then analyzed on the same basis.

7.2 Preliminaries

7.2.1 Determinantal-permanental point processes

The following set of hypothesis is of constant use.

Hypothesis 1 *The Polish space E is equipped with a Radon measure λ. The map K is an Hilbert-Schmidt operator from $L^2(E, \lambda)$ into $L^2(E, \lambda)$ which satisfies the following conditions:*

i) K is a bounded symmetric integral operator on $L^2(E, \lambda)$, with kernel $K(.,.)$, i.e., for any $x \in E$,

$$Kf(x) = \int_E K(x,y)f(y)\ d\lambda(y).$$

ii) The spectrum of K is included in $[0,\, 1[$.

iii) The map K is locally of trace class, i.e., for all compact $\Lambda \subset E$, the restriction $K_\Lambda = P_\Lambda K P_\Lambda$ of K to $L^2(\Lambda)$ is of trace class.

For a real $\alpha \in [-1, 1]$ and a compact subset $\Lambda \subset E$, the map $J_{\Lambda,\alpha}$ is defined by:

$$J_{\Lambda,\alpha} = (\mathrm{I} + \alpha K_\Lambda)^{-1} K_\Lambda,$$

so that we have:

$$(\mathrm{I} + \alpha K_\Lambda)(\mathrm{I} - \alpha J_{\Lambda,\alpha}) = \mathrm{I}\,.$$

For any compact Λ, the operator $J_{\Lambda,\alpha}$ is also a trace class operator in $L^2(\Lambda,\, \lambda)$. In the following theorem, we define α-DPPP with the three equivalent characterizations: in terms of their Laplace transforms, Janossy densities and correlation functions. The theorem is also a theorem of existence, a problem which as said above is far from being trivial.

Theorem 7.2.1 (See [37]) *Assume Hypothesis 1 is satisfied. Let $\alpha \in \mathfrak{A}$. There exists a unique probability measure $\mu_{\alpha,\, K,\, \lambda}$ on the configuration space χ such that, for any nonnegative bounded measurable function f on E with compact support, we have:*

$$\begin{aligned}\mathbb{E}_{\mu_{\alpha,\, K,\, \lambda}}\left[e^{-\int f\ d\xi}\right] &= \int_{\chi} e^{-\int f\ d\xi}\ d\mu_{\alpha,\, K,\, \lambda}(\xi)\\ &= \operatorname{Det}\left(\mathrm{I} + \alpha K[1-e^{-f}]\right)^{-\frac{1}{\alpha}}, \qquad (7.2.1)\end{aligned}$$

where $K[1-e^{-f}]$ is the bounded operator on $L^2(E)$ with kernel :

$$(K[1-e^{-f}])(x,y) = \sqrt{1-exp(-f(x))}K(x,y)\,\sqrt{1-exp(-f(y))}.$$

This means that for any integer n and any $(x_1, \cdots, x_n) \in E^n$, the correlation functions of $\mu_{\alpha,\, K,\, \lambda}$ are given by:

$$\rho_{n,\, \alpha,\, K}(x_1, \cdots,\ x_n) = det_\alpha\left(K\left(x_i,\ x_j\right)\right)_{1\le i,j\le n},$$

and for $n = 0$, $\rho_{0,\, \alpha,\, K}(\emptyset) = 1$. For any compact $\Lambda \subset E$, the operator $J_{\Lambda,\, \alpha}$ is an Hilbert-Schmidt, trace class operator, whose spectrum is included in $[0, +\infty[$. For any $n \in \mathbb{N}$, any compact $\Lambda \subset E$, and any $(x_1, \cdots, x_n) \in \Lambda^n$ the n-th Janossy density is given by:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1,\ \cdots,\ x_n) = \operatorname{Det}\left(\mathrm{I} + \alpha K_\Lambda\right)^{-1/\alpha} det_\alpha\left(J_{\Lambda,\, \alpha}(x_i,\ x_j)\right)_{1\le i,j\le n}. \qquad (7.2.2)$$

For $n = 0$, we have $j^n_{\Lambda,\alpha,K_\Lambda}(\emptyset) = \operatorname{Det}\left(I + \alpha K_\Lambda\right)^{-1/\alpha}$.

For $\alpha = -1$, such a process is called a determinantal process since we have, for any $n \ge 1$:

$$\rho_{n,-1,K}(x_1, \cdots,\ x_n) = \det(K(x_i,\ x_j))_{1\le i,j\le n}.$$

For $\alpha = 1$, such a process is called a permanental process, since we have, for any $n \ge 1$:

$$\rho_{n,1,K}(x_1, \cdots, x_n) = \sum_{\pi \epsilon \Sigma} \prod_{i=1}^{n} K(x_i,\ x_{\pi(i)}) = \operatorname{per}\left(K(x_i,\ x_j)\right)_{1\le i,j\le n}.$$

For any bounded function $g : E \to \mathbb{R}^+$, and any integral operator T of kernel $T(x,y)$, we denote by $T[g]$ the integral operator of kernel:

$$T[g](x,y) \to \sqrt{g(x)}T(x,y)\sqrt{g(y)}.$$

For calculations, it will be convenient to use the following lemma:

Lemma 7.2.1 (see [37]) *Let Λ be a compact subset of E and $f : E \to [0,+\infty)$, measurable with supp$(f) \in \Lambda$:*

$$\operatorname{Det}\left(\mathrm{I} + \alpha K_\Lambda[1-e^{-f}]\right)^{-1/\alpha} = \operatorname{Det}\left(\mathrm{I} + \alpha K_\Lambda\right)^{-1/\alpha} \operatorname{Det}\left(\mathrm{I} - \alpha J_{\Lambda,\alpha}[e^{-f}]\right)^{-1/\alpha}.$$

By differentiation into the Laplace transform, it is possible to compute moments of $\int f \, \mathrm{d}\xi$ for any deterministic f. We obtain, at the first order:

Theorem 7.2.2 (see [37]) *For any non-negative function f defined on E, we have*

$$\mathbb{E}\left[\int_\Lambda f \ d\xi\right] = \int_\Lambda f(x)K(x,\,x)\ d\lambda(x) = \operatorname{trace}(K_\Lambda[f]).$$

It is worth mentioning how the existence of α-DPPP is established. For $\alpha = -1$, there is a non trivial work (see [37, 39] and references therein) to show that the Janossy densities satisfy the positivity condition so that a point process with these densities does exist. For $\alpha = -1/m$, it is sufficient to remark from (7.2.1) that the superposition of m independent determinantal point processes of kernel K/m is an α-DPPP fo kernel K. The point is that K/m satisfies Hypothesis 1, in particular that its spectrum is strictly bounded by $1/m < 1$, since $m > 1$. For $\alpha = 2$, a 2-permanental point process is in fact a Cox process based on a Gaussian random field (see equations (6.2.4) and (6.2.5)). Thus, any $2/m$-permanental point process is the superposition of m independent 2-permanental point processes with kernel K/m.

Poisson process can be obtained formally as extreme case of 1-permanental process with a kernel K given by $K(x,\,y) = \mathbf{1}_{\{x=y\}}$. Of course, this kernel is likely to be null almost surely with respect to $\lambda \otimes \lambda$; nonetheless, it remains that replacing formally this expression in (7.2.1) yields the Laplace transform of a Poisson process of intensity λ. Another way to retrieve a Poisson process is to let α go to 0 in (7.2.1), see Theorem . With the above constructions, this means that a Poisson process can be viewed as an infinite superposition of determinantal or permanental point processes.

7.3 Quasi-invariance

In this part we show the quasi-invariance property for any α-DPPP. Let $\mathrm{Diff}_0(E)$ be the set of all diffeomorphisms from E into itself with compact support, i.e., for any $\phi \in \mathrm{Diff}_0(E)$, there exists a compact Λ outside which ϕ is the identity map. For any $\xi \in \chi$, we still denote by ϕ the map:

$$\begin{aligned} \phi \, : \, \chi &\longrightarrow \chi \\ \sum_{x\in\xi} \delta_x &\longmapsto \sum_{x\in\xi} \delta_{\phi(x)}. \end{aligned}$$

For any reference measure λ on E, λ_ϕ denotes the image measure of λ by ϕ. For $\phi \in \mathrm{Diff}_0(E)$ whose support is included in Λ, we introduce the isometry Φ,

$$\begin{aligned} \Phi \, : \, L^2(\lambda_\phi, \Lambda) &\longrightarrow L^2(\lambda, \Lambda) \\ f &\longmapsto f \circ \phi. \end{aligned}$$

Its inverse, which exists since ϕ is a diffeomorphism, is trivially defined by $f \circ \phi^{-1}$ and denoted by Φ^{-1}. Note that Φ and Φ^{-1} are isometries, i.e.,

$$\langle \Phi\psi_1, \ \Phi\psi_2 \rangle_{L^2(\lambda,\Lambda)} = \langle \psi_1, \ \psi_2 \rangle_{L^2(\lambda_\phi,\Lambda)},$$

for any ψ_1 and ψ_2 belonging to $L^2(\lambda, \Lambda)$. We also set:

$$K_\Lambda^\phi = \Phi^{-1} K_\Lambda \Phi \text{ and } J_{\Lambda,\alpha}^\phi = \Phi^{-1} J_{\Lambda,\alpha} \Phi.$$

Lemma 7.3.1 *Let λ be a Radon measure on E and K a map satisfying hypothesis 1. Let $\alpha \in \mathfrak{A}$. We have the following properties.*

a) K_Λ^ϕ and $J_{\Lambda,\alpha}^\phi$ are continuous operators from $L^2(\lambda_\phi, \Lambda)$ into $L^2(\lambda_\phi, \Lambda)$.

b) K_Λ^ϕ is of trace class and $\mathrm{trace}(K_\Lambda^\phi) = \mathrm{trace}(K_\Lambda)$.

c) $\mathrm{Det}(\mathrm{I} + \alpha K_\Lambda^\phi) = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda)$.

Proof. The first point is immediate according to the definition of an image measure. Since Φ^{-1} is an isometry, for any $(\psi_n, \, n \in \mathbb{N})$ a complete orthonormal basis of $L^2(\lambda, \Lambda)$, the family $(\Phi^{-1}\psi_n, \, , \in \mathbb{N})$ is a CONB of $L^2(\lambda_\phi, \Lambda)$. Moreover,

$$\begin{aligned} \sum_{n\geq 1} \left| \langle K_\Lambda^\phi \Phi^{-1}\psi_n, \, \Phi^{-1}\psi_n \rangle_{L^2(\lambda_\phi,\Lambda)} \right| &= \sum_{n\geq 1} \left| \langle \Phi^{-1} K \Phi \Phi^{-1}\psi_n, \, \Phi^{-1}\psi_n \rangle_{L^2(\lambda_\phi,\Lambda)} \right| \\ &= \sum_{n\geq 1} \left| \langle \Phi^{-1} K \psi_n, \, \Phi^{-1}\psi_n \rangle_{L^2(\lambda_\phi,\Lambda)} \right| \\ &= \sum_{n\geq 1} \left| \langle K \psi_n, \, \psi_n \rangle_{L^2(\lambda,\Lambda)} \right|. \end{aligned}$$

Hence, K_Λ^ϕ is of trace class and $\operatorname{trace}(K_\Lambda^\phi) = \operatorname{trace}(K_\Lambda)$. Along the same lines, we prove that $\operatorname{trace}((K_\Lambda^\phi)^n) = \operatorname{trace}(K_\Lambda^n)$ for any $n \geq 2$. According to Definition 6.2.3, the Fredholm determinant of K_Λ^ϕ is well defined and point c) follows. ■

Theorem 7.3.1 *Assume that K is a kernel operator. Then K_Λ^ϕ, as a map from $L^2(\lambda_\phi, \Lambda)$ into itself is a kernel operator whose kernel is given by $((x, y) \mapsto K_\Lambda(\phi^{-1}(x), \phi^{-1}(y)))$. An analog formula also holds for the operator $J_{\Lambda,\alpha}$.*

Proof. On the one hand, for any function f, the operator K_Λ^ϕ from $L^2(\Lambda, \lambda_\phi)$ into $L^2(\Lambda, \lambda_\phi)$ is given by :

$$K_\Lambda^\phi f(x) = \int_\Lambda K_\Lambda^\phi(x, z) f(z) d\lambda_\phi(z).$$

On the other hand, using the definition $K_\Lambda^\phi = \Phi^{-1} K_\Lambda \Phi$

$$\begin{aligned} K_\Lambda^\phi f(x) &= \Phi^{-1} K_\Lambda \Phi f(x) \\ &= \int_\Lambda K_\Lambda(\phi^{-1}(x), y) f \circ \phi(y) d\lambda(y) \\ &= \int_\Lambda K_\Lambda(\phi^{-1}(x), \phi^{-1}(z)) f(z) d\lambda_\phi(z). \end{aligned}$$

The proof is thus complete. ■

Lemma 7.3.2 *Let $\rho : E \to \mathbb{R}$ be non negative and assume that $d\lambda = \rho\ dm$ for some other Radon measure on E. Let K satisfy Hypothesis 1. Then, we have the following properties:*

1. *The map $K[\rho]$ is continuous from $L^2(m)$ into itself.*
2. *The map $K[\rho]$ is locally trace class and* $\operatorname{trace}(K_\Lambda[\rho]) = \operatorname{trace}(K_\Lambda)$.
3. *The measure $\mu_{\alpha,K,\lambda}$ is identical to the measure $\mu_{\alpha,K[\rho],m}$.*

That is to say, in some sense, we can "transfer" a part of the reference measure into the operator and vice-versa.

Proof. Remember that

$$K[\rho](x, y) = \sqrt{\rho(x)}\, K(x, y)\, \sqrt{\rho(y)}.$$

Hence

$$K_\Lambda[\rho] f(x) = \sqrt{\rho(x)} \int_\Lambda K_\Lambda(x, y) \sqrt{\rho(y)}\ \mathrm{d}\lambda(y),$$

thus

$$\int_\Lambda |K_\Lambda[\rho]f|^2 \, \mathrm{d}m = \int_\Lambda |K_\Lambda f|^2 \, \mathrm{d}\lambda,$$

and the first point follows. Consider $(\psi_n,\ n \in \mathbb{N})$, a CONB of $L^2(\lambda)$. Then $(\psi_n\sqrt{\rho}, n \in \mathbb{N})$ is a CONB of $L^2(m)$. Furthermore, we have:

$$\begin{aligned}\sum_{n\geq 1} \left|\langle K_\Lambda[\rho]\psi_n, \psi_n\rangle_{L^2(dm)}\right| &= \sum_{n\geq 1} \left|\langle K_\Lambda\sqrt{\rho}\psi_n, \sqrt{\rho}\psi_n\rangle_{L^2(dm)}\right| \\ &= \sum_{n\geq 1} \left|\langle K_\Lambda\psi_n, \psi_n\rangle_{L^2(\lambda)}\right|.\end{aligned}$$

Therefore the operator $K_\Lambda[\rho]$ is of trace class and

$$\mathrm{trace}(K_\Lambda[\rho]) = \mathrm{trace}(K_\Lambda).$$

Similarly we can prove that for any $n \geq 2$, we have $\mathrm{trace}(K_\Lambda^n[\rho]) = \mathrm{trace}(K_\Lambda^n)$. Then, using the definition of a Fredholm determinant, we have:

$$\mathrm{Det}(\mathrm{I} + \alpha K_\Lambda) = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda[\rho]).$$

The third point then follows from the characterization of $\mu_{\alpha,K[\rho],m}$ by its Laplace transform.

■

The expression $\det_\alpha J_{\Lambda,\alpha}(x_i,\ x_j)_{1\leq i,j\leq n}$ is now denoted $\det_\alpha J_{\Lambda,\alpha}(x_1, \cdots, x_n)$. For any finite random configuration $\xi = (x_1, \cdots, x_n)$, we call $J_{\Lambda,\alpha}(\xi)$ the matrix with terms $(J_{\Lambda,\alpha}(x_i,\ x_j),\ 1 \leq i,\ j \leq n)$. First, remind some results from [3] concerning Poisson measures. For any $\phi \in \mathrm{Diff}_0(E)$, we define $\phi^*\pi_\lambda$ as the image of the Poisson measure π_λ with intensity measure λ and λ_ϕ denotes the image measure of λ by ϕ.

Theorem 7.3.2 ([3]) *For any $\phi \in \mathrm{Diff}_0(E)$, and a Poisson measure π_λ with intensity λ:*

$$\phi^*\pi_\lambda = \pi_{\lambda_\phi}.$$

That is to say, for any f nonnegative and compactly supported on E:

$$\mathbb{E}_{\pi_\lambda}\left[e^{-\int f\circ\phi \ d\xi}\right] = exp\left(-\int (1 - e^{-f}) \ d\lambda_\phi\right). \tag{7.3.1}$$

We give the corresponding formula for α-determinantal measures. For any $\phi \in \mathrm{Diff}_0(E)$, we define $\phi^*\mu_{\alpha,K_\Lambda,\lambda}$ as the image of the measure $\mu_{\alpha,K_\Lambda,\lambda}$ under ϕ. We prove below that this image measure is an α-DPPP the parameters of which are explicitely known.

Theorem 7.3.3 *With the notations and hypothesis introduced above. For any $\phi \in \mathrm{Diff}_0(E)$, for any nonnegative function f on E, for any compact $\Lambda \subset E$, we have:*

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\circ\phi \ d\xi}\right] = \mathbb{E}_{\mu_{\alpha,K_\Lambda^\phi,\lambda_\phi}}\left[e^{-\int f \ d\xi}\right] \tag{7.3.2}$$
$$= \mathrm{Det}(I+\alpha K_\Lambda^\phi[1-e^{-f}])^{-1/\alpha}.$$

That is to say the image measure of $\mu_{\alpha,K,\lambda}$ by ϕ is an α-determinantal process with operator K^ϕ and reference measure λ_ϕ.

Proof. According to Theorem 7.2.1 and Theorem 7.2.1, we have for a non-negative function f:

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\circ\phi \ \mathrm{d}\xi}\right] = \mathrm{Det}\left(\mathrm{I}+\alpha K_\Lambda[1-e^{-f\circ\phi}]\right)^{-1/\alpha}$$
$$= \mathrm{Det}\left(\mathrm{I}+\alpha K_\Lambda\right)^{-1/\alpha}\mathrm{Det}\left(\mathrm{I}-\alpha J_{\Lambda,\alpha}[e^{-f\circ\phi}]\right)^{-1/\alpha}.$$

According to Theorem 6.2.1, we get

$$\begin{aligned}
&\mathrm{Det}\left(\mathrm{I}-\alpha J_{\Lambda,\alpha}[e^{-f\circ\phi}]\right)^{-1/\alpha}\\
&\quad=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n}\mathrm{det}_\alpha\, J_{\Lambda,\alpha}(x_1,\cdots,x_n)e^{-\sum_{i=1}^n f(\phi(x_i))}\,\mathrm{d}\lambda(x_1)\ldots\,\mathrm{d}\lambda(x_n)\\
&\quad=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n}\mathrm{det}_\alpha\, J^\phi_{\Lambda,\alpha}(x_1,\cdots,x_n)e^{-\sum_{i=1}^n f(x_i)}\,\mathrm{d}\lambda_\phi(x_1)\ldots\,\mathrm{d}\lambda_\phi(x_n)\\
&\quad=\mathrm{Det}\left(\mathrm{I}-\alpha J^\phi_{\Lambda,\alpha}[e^{-f}]\right)^{-1/\alpha}.
\end{aligned}$$

Since $\mathrm{Det}\left(\mathrm{I}+\alpha K_\Lambda\right) = \mathrm{Det}\left(\mathrm{I}+\alpha K_\Lambda^\phi\right)$, we have:

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\circ\phi \ \mathrm{d}\xi}\right] = \mathrm{Det}\left(\mathrm{I}+\alpha K_\Lambda^\phi[1-e^{-f}]\right)^{-1/\alpha}$$
$$= \mathbb{E}_{\mu_{\alpha,\,K_\Lambda^\phi,\,\lambda_\phi}}\left[e^{-\int f\ \mathrm{d}\xi}\right].$$

The proof is thus complete.

■

For $\alpha = 2$, Theorem 7.3.3 says that the image under ϕ of a Cox process is still a Cox process of parameters K_Λ^ϕ and λ_ϕ. Such a process can be constructed as follows: Let X be a centered Gaussian random field satisfying

(6.2.4) and (6.2.5) and let $Y(x) = X(\phi^{-1}(x))$. Then, according to Lemma 7.3.1, we have: for any compact Λ,

$$\mathbb{E}^{\mathbb{P}}\left[\int_{\Lambda} Y^2(x)\ \mathrm{d}\lambda_\phi(x)\right] = \operatorname{trace}(K^{\phi}_{\Lambda})$$

and

$$\mathbb{E}^{\mathbb{P}}\left[Y(x)Y(y)\right] = K^{\phi}(x,\, y) = K(\phi^{-1}(x),\, \phi^{-1}(y)),\ \lambda_\phi \otimes \lambda_\phi,\ \text{a.s..}$$

From Theorem 7.3.2, by conditioning with respect to X, we also have:

$$\begin{aligned}
\mathbb{E}_{\mu_{2,\,K,\,\lambda}}\left[e^{-\int f\circ\phi\ \ \mathrm{d}\xi}\right] &= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}\left[e^{-\int f\circ\phi\ \ \mathrm{d}\xi}|X\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int (1-e^{-f\circ\phi})\, X^2\ \mathrm{d}\lambda\right)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int (1-e^{-f})\, Y^2\ \mathrm{d}\lambda_\phi\right)\right].
\end{aligned}$$

Thus the two approaches (fortunately) yields the same result.

We now want to prove that $\mu_{\alpha,\, K^{\phi},\, \lambda_\phi}$ is absolutely continuous with respect to $\mu_{\alpha,\, K,\, \lambda}$ and compute the corresponding Radon-Nikodym derivative. For technical reasons, we need to assume that there exists a Jacobi formula (or change of variable formula) on the measured space $(E,\, \lambda)$. This could be done in full generality for E a manifold; for the sake of simplicity, we assume hereafter that E is a domain of some $\mathbb{R}^d$. We denote by ∇^E the usual gradient on $\mathbb{R}^d$. We also introduce a new hypothesis.

Hypothesis 2 *We suppose that the measure λ is absolutely continuous with respect to the Lebesgue measure m on E. We denote by ρ the Radon-Nikodym derivative of λ with respect to m. We furthermore assume that $\sqrt{\rho}$ is in $H^{1,2}_{loc}(K(x,\, x)\ dm(x))$, i.e., ρ is weakly differentiable and for any compact Λ in E, we have:*

$$\begin{aligned}
\infty > 2\int_{\Lambda} \|\nabla^E \sqrt{\rho(x)}\|^2 K(x,\, x)\ dm(x) \\
= \int_{\Lambda} \frac{\|\nabla^E \rho(x)\|^2}{\rho(x)} K(x,\, x)\ dm(x) \\
= \int_{\Lambda} \left(\frac{\|\nabla^E \rho(x)\|}{\rho(x)}\right)^2 K(x,\, x)\ d\lambda(x).
\end{aligned}$$

Then for any $\phi \in \operatorname{Diff}_0(E)$, λ_ϕ is absolutely continuous with respect to λ and

$$p^{\lambda}_{\phi}(x) = \frac{d\lambda_\phi(x)}{d\lambda(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)}\operatorname{Jac}(\phi)(x),$$

where $\operatorname{Jac}(\phi)(x)$ is the Jacobian of ϕ at point x.

Lemma 7.3.3 *Assume $(E,\, K,\, \lambda)$ satisfy Hypothesis 1 and 2. Let $(u_n,\, n \geq 0)$ be a sequence of nonnegative real numbers such that for any $x \in \mathbb{R}$,*

$$\sum_{n\geq 0} \frac{u_n}{n!}\ |x|^n < +\infty. \tag{7.3.3}$$

For any compact $\Lambda \subset E$, we have:

$$\mathbb{E}\left[\mu_{\alpha,\, K_\Lambda,\, \lambda}\right] \frac{u_{|\xi|}}{det_\alpha J_{\Lambda,\alpha}(\xi)} < +\infty. \tag{7.3.4}$$

As a consequence, $det_\alpha J_{\Lambda,\alpha}(\xi)$ is $\mu_{\alpha,\, K_\Lambda,\, \lambda}$ almost-surely positive.

Proof. According to Theorem 7.2.1, we have:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \cdots,\ x_n) = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha} \mathrm{det}_\alpha J_{\Lambda,\alpha}(x_1, \cdots,\ x_n),$$

hence

$$\begin{aligned}
&\mathbb{E}\left[\frac{u_{|\xi|}}{\mathrm{det}_\alpha J_{\Lambda,\alpha}(\xi)}\right] \\
&\quad = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} \frac{u_n}{\mathrm{det}_\alpha J_{\Lambda,\alpha}(x_1, \cdots,\, x_n)}\ j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \cdots,\ x_n) \otimes_{j=1}^n\ \mathrm{d}\lambda(x_j) \\
&\quad = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha} \sum_{n=0}^{+\infty} \frac{u_n}{n!} \lambda(\Lambda)^n < +\infty,
\end{aligned}$$

because λ is assumed to be a Radon measure and Λ is compact. ■

Theorem 7.3.4 *Assume $(E,\, K,\, \lambda)$ satisfy Hypothesis 1 and 2. Then, the measure $\mu_{\alpha,\, K,\, \lambda}$ is quasi-invariant with respect to the group $\mathrm{Diff}_0(E)$ and for any $\phi \in \mathrm{Diff}_0(E)$, we have then:*

$$\frac{d\phi^* \mu_{\alpha,\, K,\, \lambda}}{d\mu_{\alpha,\, K,\, \lambda}}(\xi) = \left(\prod_{x\in\xi} p^\lambda_\phi(x)\right) \frac{det_\alpha\ J^\phi_{\Lambda,\alpha}(\xi)}{det_\alpha\ J_{\Lambda,\alpha}(\xi)}. \tag{7.3.5}$$

That is to say that for any measurable nonnegative, compactly supported f on E:

$$\mathbb{E}_{\mu_{\alpha,\, K,\, \lambda}}\left[e^{-\int f\circ\phi\ d\xi}\right] = \mathbb{E}_{\mu_{\alpha,\, K,\, \lambda}}\left[e^{-\int f\ d\xi} e^{\int ln(p^\lambda_\phi)\ d\xi}\ \frac{det_\alpha\ J^\phi_\alpha(\xi)}{det_\alpha\ J_\alpha(\xi)}\right]. \tag{7.3.6}$$

Proof. Since f is compactly supported and ϕ belongs to $\mathrm{Diff}_0(E)$, there exists a compact Λ which contains both the support of f and $f \circ \phi$. According to Theorem 7.3.3 and Lemma 7.3.1, we have:

$$\begin{aligned}\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\circ\phi \ \mathrm{d}\xi}\right] &= \mathbb{E}_{\mu_{\alpha,K^\phi_\Lambda,\lambda_\phi}}\left[e^{-\int f \ \mathrm{d}\xi}\right] \\ &= \mathrm{Det}\left(\mathrm{I} + \alpha K^\phi_\Lambda\right)^{-1/\alpha}\left(\sum_{n=0}^{+\infty}\frac{1}{n!}A_n\right) \\ &= \mathrm{Det}\left(\mathrm{I} + \alpha K_\Lambda\right)^{-1/\alpha}\left(\sum_{n=0}^{+\infty}\frac{1}{n!}A_n\right)\end{aligned}$$

where for any $n \in \mathbb{N}$, the A_n are the integrals:

$$\begin{aligned}A_n &= \int_{\Lambda^n}\det{}_\alpha J^\phi_{\Lambda,\alpha}(x_1,\cdots,x_n)e^{-\sum_{i=1}^n f(x_i)} \ \mathrm{d}\lambda_\phi(x_1)\ldots \ \mathrm{d}\lambda_\phi(x_n) \\ &= \int_{\Lambda^n}\det{}_\alpha J^\phi_{\Lambda,\alpha}(x_1,\cdots,x_n)e^{-\sum_{i=1}^n f(x_i)}\prod_{i=1}^n p^\lambda_\phi(x_i) \ \mathrm{d}\lambda(x_1)\ldots \ \mathrm{d}\lambda(x_n) \\ &= \int_{\Lambda^n}\det{}_\alpha J_{\Lambda,\alpha}(x_1,\cdots,x_n)\alpha_n(x_1,\cdots,x_n) \ \mathrm{d}\lambda(x_1)\ldots \ \mathrm{d}\lambda(x_n),\end{aligned}$$

where

$$\alpha_n(x_1,\cdots,x_n) = \frac{\det_\alpha J^\phi_{\Lambda,\alpha}(x_1,\cdots,x_n)}{\det_\alpha J_{\Lambda,\alpha}(x_1,\cdots,x_n)}e^{-\sum_i f(x_i)}\prod_{i=1}^n p^\lambda_\phi(x_i).$$

Hence according to (7.2.2), we can write:

$$\begin{aligned}&\mathrm{Det}\left(\mathrm{I} + \alpha K_\Lambda\right)^{-1/\alpha}\sum_{n=0}^{+\infty}\frac{1}{n!}A_n \\ &\quad = \sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n} j^n_{\Lambda,\alpha,K_\Lambda}(x_1,\cdots,x_n)\alpha_n(x_1,\cdots,x_n) \ \mathrm{d}\lambda(x_1)\ldots \ \mathrm{d}\lambda(x_n).\end{aligned}$$

Thus, we have (7.3.6).

■

Remark 7.3.1 *Should we consider Poisson process either as a* 0*-DPPP or as an* α*-DPPP with the singular kernel mentioned above, we see that the last fraction in* (7.3.6) *reduces to* 1 *and we find the well known formula of quasi-invariance for Poisson processes (see [3]) and given in Thereom 7.3.2, that is, for a Poisson process* π_λ *with intensity measure* λ*:*

$$\mathbb{E}_{\pi_\lambda}\left[e^{-\int f\circ\phi \ d\xi}\right] = \mathbb{E}_{\pi_\lambda}\left[e^{-\int f\circ\phi \ d\xi}\right], \qquad (7.3.7)$$

where $p_\phi^\lambda = d\lambda_\phi/d\lambda$. More precisely, for an α-determinantal process with kernel $K_\Lambda(x,y) = \mathbf{1}_{\{x=y\}}$, we can check that $J_{\Lambda,\alpha}(x,y) = \mathbf{1}_{\{x=y\}}$. Plugging this into equation (7.3.6), we find (7.3.7) as expected.
On the other hand, using Theorem (7.2.1), we have that $\mu_{0,K_\Lambda,\lambda}$ is a Poisson process with intensity $K_\Lambda(x,x)d\lambda$. We can check that $J_{\Lambda,0} = K_\Lambda$. Plugging this into (7.3.5) gives:

$$\prod_{x_i\in\xi} p_\phi^\lambda(x_i)\frac{K_\Lambda^\phi(x_i,x_i)}{K_\Lambda(x_i,x_i)} = \prod_{x_i\in\xi}\frac{K_\Lambda^\phi(x_i,x_i)d\lambda(\phi^{-1})(x_i)}{K_\Lambda(x_i,x_i)d\lambda(x_i)}.$$

Then the quasi-invariance formula (7.3.6) becomes (7.3.7) for a Poisson process of intensity $K_\Lambda(x,x)d\lambda(x)$.

In the following, we define:

$$L^\phi_{\mu_{\alpha,\,K,\,\lambda}}(\xi) = \left(\prod_{x\in\xi} p_\phi^\lambda(x)\right)\frac{\det_\alpha J_\alpha^\phi(\xi)}{\det_\alpha J_\alpha(\xi)}.$$

Then formula (7.3.6) can be rewritten as:

$$\mathbb{E}_{\mu_{\alpha,\,K,\,\lambda}}\left[e^{-\int f\circ\phi\ \ \mathrm{d}\xi}\right] = \mathbb{E}_{\mu_{\alpha,\,K,\,\lambda}}\left[e^{-\int f\ \mathrm{d}\xi}\ L^\phi_{\mu_{\alpha,\,K,\,\lambda}}(\xi)\right].$$

7.4 Integration by parts formula

In this section, we prove the integration by parts formula. The proof relies on a differentiation within (7.3.6). We thus need to put a manifold structure on χ. The tangent space $T_\xi\chi$ at some $\xi\in\chi$ is given as $L^2(d\xi)$, i.e., the set of all maps V from E to $\mathbb{R}$ such that:

$$\int |V(x)|^2\ \mathrm{d}\xi(x) < \infty.$$

Note that if $\xi\in\chi_0$ then $T_\xi\chi$ can be identified as $\mathbb{R}^{|\xi|}$ with the Euclidean scalar product.

We consider $V_0(E)$ the set of all C^∞-vector fields on E with compact support. For any $v\in V_0(E)$, we construct: $\phi_t^v : E\to E$, $t\in\mathbb{R}$, where the curve, for any $x\in E$

$$t\in\mathbb{R}\to\phi_t^v(x)$$

is defined as the solution to:

$$\frac{d}{dt}\phi_t^v(x) = v(\phi_t^v(x)) \text{ and } \phi_0^v(x) = x.$$

Because $v \in V_0(E)$, there is no explosion and ϕ_t^v is well-defined for each $t \in \mathbb{R}$. The mappings $\{\phi_t^v, t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms with compact support, that is to say:

- $\forall t \in \mathbb{R}, \phi_t^v \in \mathrm{Diff}_0(E)$.
- $\forall t, s \in \mathbb{R}, \phi_t^v \circ \phi_s^v = \phi_{t+s}^v$. In particular, $(\phi_t^v)^{-1} = \phi_{-t}^v$.
- For any $T > 0$, there exists a compact K such that $\phi_t^v(x) = x$ for any $x \in K^c$, for any $|t| \leq T$.

In the following, we fix $v \in V_0(E)$. For any $\xi \in \chi$, we still denote by ϕ_t^v the map:

$$\begin{aligned} \phi_t^v \,:\, \chi &\longrightarrow \chi \\ \xi = \sum_{x\in\xi} \delta_{x_i} &\longmapsto \sum_{x\in\xi} \delta_{\phi_t^v(x)} \in \chi. \end{aligned}$$

Definition 7.4.1 *A function $F : \chi \to \mathbb{R}$ is said to be differentiable at $\xi \in \chi$ whenever for any vector field $v \in V_0(E)$, the directional derivative along the vector field v*

$$\nabla_v F(\xi) = \frac{d}{dt} F(\phi_t^v(\xi))\bigg|_{t=0}$$

is well defined.

Since ϕ_t^v does not change the number of atoms of ξ, if ξ belongs to χ_0, this notion of differentiability coincides with the usual one in $\mathbb{R}^{|\xi|}$ and

$$\nabla_v F(x_1, \cdots, x_n) = \sum_{i=1}^{n} \partial_i F(x_1, \cdots, x_n) v(x_i),$$

if $\xi = \{x_1, \cdots, x_n\}$.

In the general case, a set of test functions is defined. Following the notations from [3], for a function $F : \chi \to \mathbb{R}$ we say that $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$ if:

$$F(\xi) = f\left(\int h_1 \,\mathrm{d}\xi, \cdots, \int h_N \,\mathrm{d}\xi\right),$$

for some $N \in \mathbb{N}$, $h_1, \cdots, h_N \in \mathcal{D} = C^\infty(E)$, $f \in C_b^\infty(\mathbb{R}^N)$. Then for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$, given $v \in V_0(E)$, we have:

$$F(\phi_t^v(\xi)) = f\left(\int h_1 \circ \phi_t^v \,\mathrm{d}\xi, \cdots, \int h_N \circ \phi_t^v \,\mathrm{d}\xi\right).$$

It is then clear that the directional derivative of such F exists and that:

$$\nabla_v F(\xi) = \sum_{i=1}^{N} \partial_i f \left(\int h_1 \, \mathrm{d}\xi, \cdots, \int h_N \, \mathrm{d}\xi \right) \int \nabla_v^E h_i \, \mathrm{d}\xi.$$

The gradient ∇F of a differentiable function F is defined as a map from χ into $T\chi$ such that, for any $v \in V_0(E)$,

$$\int \nabla_x F(\xi) v(x) \, \mathrm{d}\xi(x) = \nabla_v F(\xi).$$

If $\xi \in \chi_0$ and F is differentiable at χ, then

$$\nabla_x F(\xi) = \sum_{i=1}^{|\xi|} \partial_i F(\{x_1, \cdots, x_{|\xi|}\}) \mathbf{1}_{\{x=x_i\}}.$$

If ξ belongs to χ, for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$,

$$\nabla_x F(\xi) = \sum_{i=1}^{n} \partial_i f \left(\int h_1 \, \mathrm{d}\xi, \cdots, \int h_N \, \mathrm{d}\xi \right) \nabla^E h_i(x).$$

7.4.1 Determinantal point processes

In what follows, c and κ are positive constant which may vary from line to line.

In this part, we assume $\alpha = -1$ and that Hypothesis 1 and 2 hold. We denote by $\beta^\lambda(x)$ the logarithmic derivative of λ, given by: for any x in E,

$$\beta^\lambda(x) = \frac{\nabla \rho(x)}{\rho(x)} \text{ on } \{\rho(x) > 0\},$$

and $\beta^\lambda(x) = 0$ on $\{\rho(x) = 0\}$. Then, for any vector field v on E with compact support, we denote by B_v^λ the following function on χ:

$$\begin{aligned} B_v^\lambda \, : \, \chi &\longrightarrow \mathbb{R} \\ \xi &\longmapsto B_v^\lambda(\xi) = \int_E \left(\beta^\lambda(x).v(x) + \operatorname{div}(v(x)) \right) \, \mathrm{d}\xi(x), \end{aligned}$$

where $x.y$ is the euclidean scalar product of x and y in E. If $\lambda = m$,

$$B_v^m(\xi) = \int_E \operatorname{div}(v(x)) \, \mathrm{d}\xi(x)$$

and according to Theorem 7.2.2,

$$\begin{aligned}\mathbb{E}[|B_v^m(\xi)|] &\le \int_E |\operatorname{div}(v(x))| K(x,\,x)\,\mathrm{d}\lambda(x)\\ &\le \|v\|_\infty \operatorname{trace}(K_\Lambda) < \infty,\end{aligned}$$

where Λ is a compact containing the support of v. As in [46], we now define the potential energy of a finite configuration by

$$\begin{aligned} U\,:\, \chi_0 &\longrightarrow \mathbb{R}\\ \xi &\longmapsto -\log\det J(\xi).\end{aligned}$$

Hypothesis 3 *The functional U is differentiable at every configuration $\xi \in \chi_0$. Moreover, for any $v \in V_0(E)$, there exists $c > 0$ such that for any $\xi \in \chi_0$, we have*

$$\left|\langle \nabla U(\xi),\, v\rangle_{L^2(d\xi)}\right| \le \frac{u_{|\xi|}}{\det J(\xi)}, \tag{7.4.1}$$

where $(u_n = cn^{n/2},\, n \ge 1)$ satisfy (7.3.3).

Theorem 7.4.1 *Assume that the kernel J is once differentiable with continuous derivative. Then, Hypothesis 3 is satisfied.*

Proof. Let $\xi = \{x_1, \cdots, x_n\} \in \chi_0$ and let Λ be a compact subset of E whose interior contains ξ. Since $J(.,.)$ is differentiable

$$(y_1, \cdots, y_n) \longmapsto -\log\det\,(J(y_i, y_k),\, 1 \le i, k \le n)$$

is differentiable. The chain rule formula implies that

$$t \longmapsto \log\det\,(J(\phi_t^v(x_i),\, \phi_t^v(x_k)),\, 1 \le i, k \le n)$$

is differentiable and its differential is equal to

$$\frac{1}{\det J(\phi_t^v(\xi))}\operatorname{trace}\left(\operatorname{Adj}(J(\phi_t^v(x_i),\, \phi_t^v(x_k)))\left(E_t^v(\frac{\partial J(\xi)}{\partial x})_t + (\frac{\partial J(\xi)}{\partial y})_t E_t^v\right)\right),$$

where $(\frac{\partial J(\xi)}{\partial x})_t$ is the matrix with terms $\left(\frac{\partial J_\Lambda}{\partial x}(\phi_t^v(x_i), \phi_t^v(x_j))\right)_{x_i,\, x_j \in \xi}$, $(\frac{\partial J(\xi)}{\partial y})_t$ is the matrix with terms $\left(\frac{\partial J_\Lambda}{\partial y}(\phi_t^v(x_i), \phi_t^v(x_j))\right)_{x_i,\, x_j \in \xi}$, and E_t^v is the diagonal matrix with terms $(v(\phi_t^v(x_i)))_{x_i \in \xi}$. For $t = 0$, this reduces to

$$\begin{aligned}&\left|\langle \nabla U(\xi),\, v\rangle_{L^2(d\xi)}\right| =\\ &\qquad \frac{1}{\det J(\xi)}\operatorname{trace}\left(\operatorname{Adj}(J(\xi))\left(E_0^v(\frac{\partial J(\xi)}{\partial x})_0 + (\frac{\partial J(\xi)}{\partial y})_0 E_0^v\right)\right).\end{aligned}$$

Since J is continuous and Λ is compact,

$$\|\frac{\partial J}{\partial y}(\xi)\|_{HS} \leq |\xi|\|J\|_\infty \text{ and } \|E_0^v(\xi)\|_{HS} \leq |\xi|^{1/2}\|v\|_\infty.$$

Hence, there exists c independent of ξ such that

$$\left|\langle \nabla U(\xi),\, v\rangle_{L^2(d\xi)}\right| \leq c\,|\xi|^2 \frac{1}{\det J(\xi)}\,|\operatorname{trace}(\operatorname{Adj}(J(\xi)))|.$$

From [20, page 1021], we know that for any $n \times n$ matrix A, for any x and y in $\mathbb{R}^n$, we have

$$|(\operatorname{Adj} A)x.y| \leq \|y\|\|A\|_{HS}^{n-1}(n-1)^{-(n-1)/2}.$$

It follows that

$$|\operatorname{trace}(\operatorname{Adj} A)| = |\sum_{j=1}^n (\operatorname{Adj} A)e_j.e_j| \leq n\|A\|_{HS}^{n-1}(n-1)^{-(n-1)/2},$$

where $(e_j,\, j = 1, \cdots, n)$ is the canonical basis of $\mathbb{R}^n$. Since J is bounded, $\|J(\xi)\|_{HS} \leq |\xi|\|J\|_\infty$, hence there exists c independent of ξ such that

$$\left|\langle \nabla U(\xi),\, v\rangle_{L^2(d\xi)}\right| \leq \frac{c}{\det J(\xi)}|\xi|^{|\xi|/2}.$$

The proof is thus complete. ■

Corollary 1 *Assume that hypothesis 3 holds. For any $v \in V_0(E)$, for any $\xi \in \chi_0$, the function*

$$t \longmapsto H_t(\xi) = \frac{det\, J(\phi_t^v(\xi))}{det\, J(\xi)}$$

is differentiable and

$$\sup_{|t|\leq T}\left|\frac{dH_t(\xi)}{dt}\right| \leq \frac{u_{|\xi|}}{det\, J(\xi)},$$

where $(u_n,\, n \geq 0)$ satisfy (7.3.3).

Proof. According to Hypothesis 3, the function $(t \mapsto U(\phi_t^v(\xi)))$ is differentiable and

$$\frac{dU(\phi_t^v(\xi))}{dt} = \langle \nabla U(\phi_t^v(\xi)),\ v\rangle_{L^2(d\phi_t^v(\xi))}. \tag{7.4.2}$$

For any t, ϕ_t^v is a diffeomorphism hence, Theorem 7.3.4 applied to ϕ_t^v and ϕ_{-t}^v implies that $\mu_{-1,\, K^{\phi_t^v}, \lambda_{\phi_t^v}}$ and $\mu_{-1,\, K,\, \lambda}$ are equivalent measure. According to Lemma 7.3.3, for any t, $\det J^{\phi_t^v}(\xi)$ is $\mu_{-1,\, K^{\phi_t^v},\lambda_{\phi_t^v}}$-a.s. positive hence it is also $\mu_{-1,\,K,\,\lambda}$-a.s. positive. Since for any $\xi \in \chi_0$,

$$t \mapsto \det J^{\phi_t^v}(\xi) = \exp(-U(\phi_t^v(\xi)))$$

is continuous, it follows that there exists a set of full $\mu_{-1,\,K,\,\lambda}$ measure on which $\det J^{\phi_t^v}(\xi) > 0$ for any $|t| \le T$, for any ξ. Furthermore,

$$\frac{dH_t(\xi)}{dt} = -\frac{\det J(\phi_t^v(\xi))}{\det J(\xi)}\ \frac{dU(\phi_t^v(\xi))}{dt}.$$

In view of (7.4.2) and of Hypothesis 3, this means that

$$\begin{aligned}\sup_{|t|\le T}\left|\frac{dH_t(\xi)}{dt}\right| &\le \frac{\det J(\phi_t^v(\xi))}{\det J(\xi)}\ \frac{u_{|\xi|}}{\det J(\phi_t^v(\xi))}\\ &= \frac{u_{|\xi|}}{\det J(\xi)},\end{aligned}$$

since $\phi_t^v(\xi)$ has the number of atoms as ξ. ■

Lemma 7.4.1 *Assume that $\lambda = m$ and set*

$$P_t(\xi) = \prod_{x\in\xi} p_{\phi_t^v}(x) = \prod_{x\in\xi} \operatorname{Jac}\phi_t^v(x).$$

For any $v \in \operatorname{Diff}_0(E)$, for any configuration $\xi \in \chi$, P is differentiable with respect to t and we have

$$\frac{d\log P_t}{dt}(\xi) = \int \left(\operatorname{div} v - \int_0^t \nabla^E \operatorname{div} v \circ \eta_{r,t}.\ v(\eta_{r,t})\ dr\right) d\xi,$$

where for any $r \le t$, $x \to \eta_{r,t}(x)$ is the diffeomorphism of E which satisfies:

$$\eta_{r,t}(x) = x - \int_r^t v(\eta_{s,t}(x))\ ds.$$

In particular for $t = 0$, we have:

$$\frac{d}{dt}\left(\prod_{x\in\xi} p^{\lambda}_{\phi_t^v}(x)\right)\Bigg|_{t=0} = B_v^m(\xi). \tag{7.4.3}$$

Moreover, there exists $c > 0$ and $\kappa > 0$ such that for any $\xi \in \chi_0$,

$$\sup_{t\le T}\left|\frac{dP_t}{dt}(\xi)\right| \le c e^{\kappa|\xi|}. \tag{7.4.4}$$

Proof. Introduce, for any $s \leq t$, $x \longmapsto \eta_{s,t}(x)$, the diffeomorphism of E which satisfies:

$$\eta_{s,t}(x) = x - \int_s^t v(\eta_{r,t}(x))\ \mathrm{d}r.$$

As a comparison, we remind that $\phi_t^v(x) = x + \int_0^t v(\phi_s^v(x))\ \mathrm{d}s$. It is well-known that the diffeomorphism $x \longmapsto \eta_{0,t}(x)$ is the inverse of $x \longmapsto \phi_t^v(x)$. Then using [43], we have:

$$\operatorname{Jac}\phi_t^v(x) = \frac{d(\phi_t^v)^* m(x)}{dm(x)} = \exp\left(\int_0^t \mathrm{div}\ \ v \circ \eta_{r,t}(x)\ \mathrm{d}r\right), \tag{7.4.5}$$

and:

$$\prod_{x\in\xi} \operatorname{Jac}\phi_t^v(x) = \exp\left(\sum_{x\in\xi}\int_0^t \mathrm{div}\ \ v \circ \eta_{r,t}(x)\ \mathrm{d}r\right).$$

Hence, we have:

$$\begin{aligned}\sum_{x\in\xi}\frac{d}{dt}\log\operatorname{Jac}\phi_t^v(x) &= \sum_{x\in\xi}\frac{d}{dt}\int_0^t \mathrm{div}\ \ v\circ\eta_{r,t}(x)\ \mathrm{d}r\\ &= \sum_{x\in\xi}\mathrm{div}\, v(x) - \int_0^t \nabla^E \mathrm{div}\, v\circ\eta_{r,t}(x).v(\eta_{r,t}(x))\ \mathrm{d}r.\end{aligned}$$

The first and second point follow easily. Now, v is assumed to have bounded derivatives of any order, hence for any $\xi \in \chi_0$,

$$\left|\frac{d\log P_t}{dt}(\xi)\right| \leq c|\xi|, \tag{7.4.6}$$

where c does not depend neither from t nor ξ. According to (7.4.5), there exists $\kappa > 0$ such that for any $\xi \in \chi_0$, we have:

$$|P_t(\xi)| \leq \exp(\kappa|\xi|). \tag{7.4.7}$$

Thus, combining (7.4.6) and (7.4.7), we get (7.4.4). ■ We are now in position to prove the main result of this section.

Theorem 7.4.2 *Assume (E, K, λ) satisfy Hypothesis 1, 2 and 3, let $\alpha = -1$. Let F and G belong to $\mathcal{F}C_b^\infty$. For any compact Λ, we have:*

$$\begin{aligned}\int_{\chi_\Lambda}\nabla_v F(\xi)G(\xi)\ d\mu_{-1,K_\Lambda,\lambda}(\xi) &= -\int_{\chi_\Lambda} F(\xi)\nabla_v G(\xi)\ d\mu_{-1,K_\Lambda,\lambda}(\xi)\\ &+ \int_{\chi_\Lambda} F(\xi)G(\xi)\left(B_v^\lambda(\xi) + \nabla_v U(\xi)\right)\ d\mu_{-1,K_\Lambda,\lambda}(\xi).\end{aligned} \tag{7.4.8}$$

Proof. In view of Lemma 7.3.2, we can replace J by $J[\rho]$ and assume $\lambda = m$, i.e., λ is the Lebesgue measure. Note that

$$B_v^m(\xi) = \int \operatorname{div} v(x) \, \mathrm{d}\xi(x).$$

Let Λ be a fixed compact set in E, remember that $\chi_\Lambda \subset \chi_0$. Let M be an integer and $\chi^M = \{\xi \in \chi_0, \ |\xi| \leq M\}$. It is crucial to note that χ^M is invariant by any $\phi \in \operatorname{Diff}_0(E)$. On the one hand, by dominated convergence, we have:

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\chi^M} F(\phi_t^v(\xi))G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi)\right)\bigg|_{t=0} \\
&\qquad = \int_{\chi^M} \frac{d}{dt}\left(F(\phi_t^v(\xi))\right)\bigg|_{t=0} G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi) \\
&\qquad = \int_{\chi^M} \nabla_v F(\xi) G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi).
\end{aligned}$$

On the other hand, we know from (7.3.6) that

$$\begin{aligned}
&\int_{\chi^M} F(\phi_t^v(\xi))G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi) \\
&\quad = \int_{\chi_\Lambda} F(\phi_t^v(\xi))G(\xi)\mathbf{1}_{\{|\xi|\leq M\}} \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi) \\
&\quad = \int_{\chi_\Lambda} F(\xi)G(\phi_{-t}^v(\xi))\mathbf{1}_{\{|\phi_{-t}^v(\xi)|\leq M\}} \, \mathrm{d}\mu_{-1,K_\Lambda^{\phi_t^v}[\rho],m_{\phi_t^v}}(\xi) \qquad (7.4.9)\\
&\quad = \int_{\chi_\Lambda} F(\xi)G(\phi_{-t}^v(\xi))\mathbf{1}_{\{|\xi|\leq M\}} L_{-1,\,K[\rho],\,\lambda}^{\phi_t^v}(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi).
\end{aligned}$$

According to Corollary 1 and Lemma 7.4.1, the function $(t \mapsto L_{-1,\,K[\rho],\,\lambda}^{\phi_t^v}(\xi))$ is differentiable and there exists c such that:

$$\sup_{t\leq T}\left|\frac{dL_{-1,\,K[\rho],\,\lambda}^{\phi_t^v}}{dt}(\xi)\right| \leq \frac{u_{|\xi|}}{\det J(\xi)},$$

where $(u_n, n \geq 0)$ satisfy (7.3.3).
Lemma 7.3.3 implies that the right-hand-side of the last inequality is integrable with respect to $\mu_{-1,\,K_\Lambda,\,\lambda}$, thus, we can differentiate inside the expectations in (7.4.9) and we obtain:

$$\begin{aligned}
&\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi) \mathbf{1}_{\{|\xi|\leq M\}} \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi) \\
&= \int_{\chi_\Lambda} F(\xi)\left(-\nabla_v G(\xi) + G(\xi)\left(B_v^m(\xi) + \nabla_v U(\xi)\right)\right)\mathbf{1}_{\{|\xi|\leq M\}} \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi).
\end{aligned}$$

According to Hypothesis 3 and Lemma 7.3.3, by dominated convergence, we have:

$$\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi)$$
$$= \int_{\chi_\Lambda} F(\xi) \left(-\nabla_v G(\xi) + G(\xi) \left(B_v^m(\xi) + \nabla_v U(\xi)\right)\right) \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi).$$

Now, we remark that

$$\begin{aligned}
\nabla_v U[\rho](\xi) &= \nabla_v \log \det J[\rho](\xi) \\
&= \nabla_v \log \left(\prod_{x \in \xi} \rho(x) \det J(\xi) \right) \\
&= \nabla_v \int \log \rho(x) \, \mathrm{d}\xi(x) + \nabla_v U(\xi) \\
&= \int \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) + \nabla_v U(\xi).
\end{aligned}$$

Moreover, we have

$$B_v^m(\xi) + \int_\Lambda \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) = B_v^\lambda(\xi),$$

and in view of Theorem 7.2.2,

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_\Lambda \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) \right| \right]^2 \\
&\qquad \leq \mathbb{E}\left[\int_\Lambda \left(\frac{\|\nabla^E \rho(x)\|}{\rho(x)} \right)^2 \, \mathrm{d}\xi(x) \right] \mathbb{E}\left[\int_\Lambda |v(x)|^2 \, \mathrm{d}\xi(x) \right] \\
&\qquad \leq \|v\|_\infty^2 \, \mathrm{trace}(K_\Lambda) \int_\Lambda \left(\frac{\|\nabla^E \rho(x)\|}{\rho(x)} \right)^2 K(x,\, x) \rho(x) \, \mathrm{d}m(x).
\end{aligned}$$

Then, Hypothesis 2 implies that B_v^λ is integrable and we get (7.4.8) in the general case.

Remark 7.4.1 *As mentioned before, if K has kernel $K(x,y) = \mathbf{1}_{\{x=y\}}$, then μ is a Poisson measure with intensity λ. And, if $K(x,y) = \mathbf{1}_{\{x=y\}}$, then $\nabla U^{J_\Lambda} \equiv 0$. We can check that plugging $\nabla U^{J_\Lambda} \equiv 0$ in (7.4.8), the last term vanishes and it becomes the integration by parts formula for a Poisson process.*

■

7.4.2 α-determinantal point processes

We now prove the integration by parts formula for α-determinantal point processes where $\alpha = -1/s$ for s integer greater than 2. In principle, we could follow the previous lines of proof modifying the definition of U as

$$U(\xi) = -\log \det_\alpha J_\alpha(\xi)$$

and assuming that Hypothesis 3 is still valid. Unfortunately, there is no (simple) analog of Theorem 7.4.1 since there is no rule to differentiate an α-determinant and control its derivative.

We already saw that such an α-DPPP can be obtained as the superposition of s determinantal processes of kernel K/s.

Let $(E_1,\, \lambda_1,\, K_1), \cdots, (E_s,\, \lambda_s,\, K_s)$ be s Polish spaces each equipped with a Radon measure and s linear operators satisfying Hypothesis 1 on their respective space. We set

$$E = \cup_{i=1}^{s} \{i\} \times E_i,$$

that is to say E is the disjoint union of the E_i's, often denoted as $\sqcup_{i=1}^{s} E_i$. An element of E is thus a couple $(i,\, x)$ where x belongs to E_i for any $i \in \{1, \cdots, s\}$. On the Polish space E, we put the measure λ defined by

$$\int_E f(i,\, x) \, \mathrm{d}\lambda(i,\ x) = \int_E f(i,\, x) \, \mathrm{d}\lambda_i(x).$$

We also define K as

$$Kf(i,\, x) = \int_{E_i} K_i(x,\, y) f(y) \, \mathrm{d}\lambda_i(y).$$

A compact set in E is of the form $\Lambda = \cup_{i=1}^{s} \{i\} \times \Lambda_i$ where Λ_i is a compact set of E_i hence

$$K_\Lambda f(i,\, x) = \int_{\Lambda_i} K_i(x,\, y) f(y) \, \mathrm{d}\lambda_i(y).$$

This means that K is a kernel operator the kernel of which is given by:

$$K((i,\, x),\, (j,\, y)) = K_i(x,\, y) \mathbf{1}_{\{i=j\}}. \tag{7.4.10}$$

In particular, for $\xi = ((i_l,\, x_l),\ l = 1, \cdots, n)$, we have

$$\det K(\xi) = \prod_{j=1}^{s} \det K(\xi_j)$$

where $\xi_j = \{x,\ (j,\, x) \in \xi\}$.

It is straightforward that K is symmetric and locally of trace class. Moreover, its spectrum is equal to the union of the spectra of the K_i's. For, if ψ is such that $K\psi = \alpha\psi$ then $\psi(i, .)$ is an eigenvector of K_i and thus α belongs to the spectrum of K_i. In the reverse direction, if ψ is an eigenvector of K_i associated to the eigenvalue α then the function

$$f(j,\, x) = \psi(x)\mathbf{1}_{\{i=j\}}$$

is square integrable with respect to λ and is an eigenvector of K for the eigenvalue α. If we assume furthermore that each of the E_i's is a subset of $\mathbb{R}^d$, we can define the gradient on E as

$$\nabla^E f(i,\, x) = \nabla^{E_i} f(i,\, x).$$

Now χ_E is the set of locally finite point measures of the form

$$\xi = \sum_j \delta_{(i_j,\, x_j)}.$$

With these notations, it is clear that Hypothesis 1, 2 and 3 are satisfied provided they are satisfied for each index i. Thus (7.4.8) is satisfied.

Now take $E_1 = \ldots = E_s$, $\lambda_1 = \ldots = \lambda_s$ and $K_1 = \ldots = K_s$. We introduce the map Θ defined as:

$$\begin{aligned} \Theta\, :\, E &\longrightarrow E_1 \\ (i,\, x) &\longmapsto x. \end{aligned}$$

Consistently with earlier defined notations, we still denote by Θ the map

$$\begin{aligned} \Theta\, :\, \chi_E &\longrightarrow \chi_{E_1} \\ \xi &\longmapsto \sum_{(j,\, x) \in \xi} \delta_x. \end{aligned}$$

Then, according to what has been said above, $\mu_{-1/s,\, sK_1,\, \lambda_1}$ is the image measure of $\mu_{-1,\, K,\, \lambda}$ by the map Θ. Set

$$\xi_n = \sum_{(i,\, x) \in \xi} \delta_x \mathbf{1}_{\{i=n\}}.$$

The reciprocal problem, interesting in its own sake and useful for the sequel, is to determine the conditional distribution of ξ_1 given $\Theta\xi$.

Theorem 7.4.3 *Let s be an integer strictly greater than 1, for F non-negative or bounded, for any Λ compact subset of E,*

$$\mathbb{E}\left[F(\xi_1)\,|\,\Theta\xi\right] = \sum_{\eta\subset\Theta\xi} F(\eta)\times\binom{|\Theta\xi|}{|\eta|}\frac{j_{\beta,\,(s-1)K_{1,\Lambda},\,\lambda_1}(\Theta\xi\backslash\eta)\ j_{-1,\,K_{1,\Lambda},\,\lambda_1}(\eta)}{j_{\alpha,sK_{1,\Lambda},\,\lambda_1}(\Theta\xi)}, \tag{7.4.11}$$

where $\beta = -1/(s-1)$. Note that (7.4.11) also holds for $s = 1$ with the convention that $j_{\beta,0}(\eta) = 0$ for $\eta \neq \emptyset$ and $j_{\beta,0}(\emptyset) = 1$, which is analog to the usual convention $0^0 = 1$.

Proof. Let $\zeta = \xi_2 \cup \ldots \cup \xi_s$, we known that ζ is distributed as $\mu_{-\beta,\,-K_1/\beta,\,\lambda_1}$. Consider Ξ, the map

$$\begin{aligned}\Xi\,:\,\chi_{E_1}\times\chi_{E_1} &\longrightarrow \chi_{E_1}\times\chi_{E_1}\\ (\eta_1,\,\eta_2) &\longmapsto (\eta_1,\,\eta_1\cup\eta_2).\end{aligned}$$

By construction, the joint distribution of $\Xi(\xi_1,\,\zeta)$ is the same as the distribution of $(\xi_1,\,\Theta\xi)$. For any $\eta\subset\Theta\xi\in\chi_0$, we set:

$$R(\eta,\,\Theta\xi) = \binom{|\Theta\xi|}{|\eta|}\frac{j_{\beta,\,(s-1)K_{1,\Lambda},\,\lambda_1}(\Theta\xi\backslash\eta)\ j_{-1,\,K_{1,\Lambda},\,\lambda_1}(\eta)}{j_{\alpha,\,sK_{1,\Lambda},\,\lambda_1}(\Theta\xi)}.$$

Hence, for any F and G bounded, we have

$$\begin{aligned}
&\mathbb{E}\left[F(\xi_1)G(\Theta\xi)\right] = \mathbb{E}\left[(F\otimes G)\circ\Xi(\xi_1,\,\zeta)\right]\\
&\quad = \sum_{j,\,k=0}^{\infty}\frac{1}{j!}\frac{1}{k!}\int_{\Lambda^j\times\Lambda^k} F(\{x_1,\cdots,x_j\})G(\{x_1,\cdots,x_j\}\cup\{y_1,\cdots,y_k\})\\
&\quad\times j_{-1,\,K_{1,\Lambda},\,\lambda_1}(x_1,\cdots,\,x_j)\ j_{\beta,\,(s-1)K_{1,\Lambda},\,\lambda_1}(y_1,\cdots,\,y_k)\ \mathrm{d}\lambda_1(x_1)\ldots\ \mathrm{d}\lambda_1(y_k)\\
&= \sum_{j,\,k=0}^{\infty}\frac{1}{(k+j)!}\int_{\Lambda^j\times\Lambda^k} F(\{x_1,\cdots,x_j\})(GR)(\{x_1,\cdots,x_j\}\cup\{y_1,\cdots,y_k\})\\
&\qquad\qquad\times j_{\alpha,sK_{1,\Lambda},\,\lambda_1}(x_1,\cdots,x_j,\,y_1,\cdots,\,y_k)\ \mathrm{d}\lambda_1(x_1)\ldots\ \mathrm{d}\lambda_1(y_k)\\
&= \sum_{m=0}^{\infty}\frac{1}{m!}\int_{\Lambda^m}\left(\sum_{j\le m}F(\{x_1,\cdots,\,x_j\})R(\{x_1,\cdots,\,x_j\},\,\{x_1,\cdots,\,x_m\})\right)\\
&\qquad\times G(\{x_1,\cdots,\,x_m\})\,j_{\alpha,sK_{1,\Lambda},\,\lambda_1}(x_1,\cdots,\,x_m)\ \mathrm{d}\lambda_1(x_1)\ldots\ \mathrm{d}\lambda_1(x_m)\\
&\qquad\qquad = \int_{\chi_{E_1}}\left(\sum_{\eta\subset\omega}F(\eta)R(\eta,\,\omega)\right)G(\omega)\ \mathrm{d}\mu_{\alpha,\,sK_1,\,\lambda_1}(\omega).
\end{aligned}$$

The proof is thus complete. ■ This formula can be understood by looking at the extreme case of Poisson process. Assume that $\Theta\xi$ is distributed according to a Poisson process of intensity λ dm. Then, ξ_1 is a Poisson process of intensity $s^{-1}\lambda$ dm and ζ also is a Poisson process of intensity $(1-s^{-1})\lambda$ dm. The couple $(\xi_1,\, \Theta\xi)$ can then be constructed by random thinning of $\Theta\xi$: Keep each point of $\Theta\xi$ independently of the others, with probability $1/s$; the remaining points will be distributed as ξ_1. The conditional expectation of a functional $F(\xi_1)$ given $\Theta\xi$ is then the sum of the values of F taken for each realization of a thinning multiplied by the probability of each thinned configuration. Since $|\Theta\xi|$ is assumed to be known, the atoms of $\Theta\xi$ are independent and identically dispatched along E, hence the probability to obtain a specific configuration is equal to the probability that a random variable binomially distributed of parameters $|\Theta\xi|$ and $1/s$, is equal to the cardinal of the configuration. This means that

$$\mathbb{E}\left[F(\xi_1)\,|\,\Theta\xi\right] = \sum_{\eta\subset\Theta\xi} F(\eta)\times\binom{|\Theta\xi|}{|\eta|}\left(\frac{1}{s}\right)^{|\eta|}\left(1-\frac{1}{s}\right)^{|\Theta\xi|-|\eta|}.$$

This corresponds to (7.4.11) for $\alpha = 0$. As a consequence, (7.4.11) can be read as a generalization of this procedure where the points cannot be drawn independently and with equal probability because of the correlation structure.

For h any map from E_1 into E_1, we define $h^{\sqcup}$ by

$$\begin{aligned} h^{\sqcup}\,:\, E &\longrightarrow E\\ (i,\,x) &\longmapsto (i,\,h(x)). \end{aligned}$$

With this notation at hand, for v in $V_0(E_1)$, $(\phi_t^v)^{\sqcup}$ is the solution of the equations:

$$\mathrm{d}(\phi_t^v)^{\sqcup}(i,\,x) = v^{\sqcup}((\phi_t^v)^{\sqcup}(i,\,x)),\ 1\le i\le m.$$

Note that we only consider a restricted set of perturbations of configurations in the sense that we move atoms on each "layers" without "crossing": By the action of $(\phi_t^v)^{\sqcup}$, an atom of the form $(i,\,x)$ is moved into an atom of the form $(i,\,y)$, leaving its first coordinate untouched.

Theorem 7.4.4 *Assume that $(E_1,\,K_1,\lambda_1)$ satisfy Hypothesis 1, 2 and 3. Let $s=-1/\alpha$ be an integer greater than 1. For F and G cylindrical functions, for $v\in V_0(E_1)$, we have:*

$$\begin{aligned}\int_{\chi_\Lambda}\nabla_v F(\omega)G(\omega)\ d\mu_{\alpha,\,sK_{1,\Lambda},\lambda_1}(\omega) &= -\int_{\chi_\Lambda}F(\omega)\nabla_v G(\omega)\ d\mu_{\alpha,\,K_{1,\,\Lambda},\lambda_1}(\omega)\\ &+\frac{1}{|\alpha|}\int_{\chi_\Lambda}F(\omega)G(\omega)\left(\sum_{\eta\subset\omega}(B_v^{\lambda_1}(\eta)+\nabla_v U(\eta))R(\eta,\,\omega)\right)\ d\mu_{\alpha,\,sK_{1,\Lambda},\lambda_1}(\omega).\end{aligned}$$

Proof. We first apply (7.4.8) to the process $\xi = (\xi_1, \cdots, \xi_s)$. Remember that $\Theta\xi$ is equal to $\xi_1 \cup \ldots \cup \xi_s$. A cylindrical function of $\Theta\xi$ is a function of the form:

$$H(\Theta\xi) = f(\int h_1 \, \mathrm{d}\Theta\xi, \cdots, \int h_N \, \mathrm{d}\Theta\xi)$$

where $h_1, \cdots, h_N \in \mathcal{D} = C^\infty(E_1)$, $f \in C_b^\infty(\mathbb{R}^N)$. Such a functional can be written as $F \circ \Theta(\xi)$ where F is a cylindrical function of ξ. Moreover, for $v \in V_0(E_1)$,

$$\begin{aligned}
\nabla_v H(\Theta\xi) &= \lim_{t\to 0} \frac{1}{t} \left(H(\phi_t^v(\Theta\xi) - H(\Theta\xi)\right) \\
&= \lim_{t\to 0} \frac{1}{t} \left(F(\Theta(\phi_t^v)^\sqcup(\xi) - F(\Theta\xi)\right) \\
&= \nabla_{v^\sqcup} F(\Theta\xi). \qquad (7.4.12)
\end{aligned}$$

In view of (7.4.10),

$$U(\xi) = -\log\det J(\xi_1, \cdots, \xi_s) = \sum_{j=1}^{s} U(\xi_j). \qquad (7.4.13)$$

Analyzing the proof of (7.4.8), we see that the intrinsic definition of B_v^λ is

$$B_v^\lambda(\xi) = \int \mathrm{div}_\lambda(v) \, \mathrm{d}\xi$$

where

$$\mathrm{div}_\lambda(v)(x) = \frac{d}{dt}\left(\frac{d(\phi_t^v)^*\lambda}{\mathrm{d}\lambda}(x)\right)\Bigg|_{t=0}.$$

In view of (7.4.12), we only need to consider flows on E associated to vector fields of the form $v^\sqcup$ for $v \in V_0(E_1)$. Hence,

$$B_{v^\sqcup}^\lambda(\xi) = \sum_{j=1}^{s} B_v^{\lambda_j}(\xi_j). \qquad (7.4.14)$$

It follows from the previous considerations that:

$$\begin{aligned}
\int_{\chi_{\Lambda^\sqcup}} \nabla_{v^\sqcup} F(\Theta\xi) G(\Theta\xi) \, \mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi) &= -\int_{\chi_{\Lambda^\sqcup}} F(\Theta\xi) \nabla_{v^\sqcup} G(\Theta\xi) \, \mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi) \\
&+ \int_{\chi_{\Lambda^\sqcup}} F(\Theta\xi) G(\Theta\xi) \left(B_{v^\sqcup}^\lambda(\xi) + \nabla_{v^\sqcup} U(\xi)\right) \, \mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi)
\end{aligned}$$

where $\Lambda^{\sqcup} = \cup_{j=1}^{s}\{i\} \times \Lambda$. Since the ξ_j's are independent and identically distributed, according to (7.4.13) and (7.4.14), we have

$$\begin{aligned}\mathbb{E}\left[B_{v^{\sqcup}}^{\lambda}(\xi) + \nabla_{v^{\sqcup}}U(\xi) \,\middle|\, \Theta\xi\right] &= s\mathbb{E}\left[B_{v}^{\lambda_1}(\xi_1) + \nabla_{v}U(\xi_1) \,\middle|\, \Theta\xi\right] \\ &= -\frac{1}{\alpha}\sum_{\eta\subset\Theta\xi}(B_v^{\lambda_1}(\eta) + \nabla_v U(\eta))R(\eta,\, \Theta\xi).\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}\int_{\chi_\Lambda} \nabla_v F(\omega)G(\omega)\ \mathrm{d}\mu_{\alpha,\, sK_{1,\Lambda},\lambda_1}(\omega) &= -\int_{\chi_\Lambda} F(\omega)\nabla_v G(\omega)\ \mathrm{d}\mu_{\alpha,\, K_{1,\,\Lambda},\lambda_1}(\omega) \\ -\frac{1}{\alpha}\int_{\chi_\Lambda} F(\omega)G(\omega)&\left(\sum_{\eta\subset\omega}(B_v^{\lambda_1}(\eta) + \nabla_v U(\eta))R(\eta,\, \omega)\right)\ \mathrm{d}\mu_{\alpha,\, sK_{1,\Lambda},\lambda_1}(\omega).\end{aligned}$$

The proof is thus complete. ■

7.4.3 α-permanental point processes

For permanental point processes, we begin with the situation where $\alpha = 1$. In this case,

$$j_{1,\, K_\Lambda,\, \lambda}(\{x_1, \cdots, x_n\}) = \mathrm{Det}(I + K_\Lambda)^{-1}\mathrm{per}(J(x_i,\, x_j),\, 1 \le i, j \le n).$$

We aim to follow the lines of proof of Theorem 7.4.2, for, we need some preliminary considerations.

For any integer n, let $D[n]$ be the set of partitions of $\{1, \cdots, n\}$. The cardinal of $D[n]$ is known to be the n-th Bell number (see [1]), denoted by $\mathfrak{B}_n$ and which can be computed by their exponential generating function: for any real x,

$$\sum_{n=0}^{\infty}\mathfrak{B}_n\frac{x^n}{n!} = e^{e^x} - 1. \tag{7.4.15}$$

For an $n \times n$ matrix $A = (a_{ij},\, 1 \le i,\, j \le n)$ and for τ a subset of $\{1, \cdots, n\}$, we denote by $A[\tau]$ the matrix $(a_{ij},\, i \in \tau,\, j \in \tau)$. For a partition σ of $\{1, \cdots, n\}$, $\iota(\sigma)$ is the number of non-empty parts of σ. This means that $\sigma = (\tau_1, \cdots, \tau_{\iota(\sigma)})$, where the τ_i's are disjoint subsets of $\{1, \cdots, n\}$ whose union is exactly $\{1, \cdots, n\}$. Then, we set

$$\det A[\sigma] = \prod_{j=1}^{\iota(\sigma)}\det J[\tau_j].$$

It is proved in [16, Corollary 1.7] that

$$\operatorname{per} A = \sum_{\sigma \in D[n]} (-1)^{n+\iota(\sigma)} \det\, A[\sigma]. \tag{7.4.16}$$

We slightly change the definition of the potential energy of a finite configuration as

$$\begin{aligned} U\,:\,\chi_0 &\longrightarrow \mathbb{R} \\ \xi &\longmapsto -\log \operatorname{per}\, J(\xi). \end{aligned}$$

A new hypothesis then arises:

Hypothesis 4 *The functional U is differentiable at every configuration $\xi \in \chi_0$. Moreover, for any $v \in V_0(E)$, there exists $(u_n,\, n \geq 1)$ a sequence of nonnegative real as in Lemma 7.3.3 such that for any $\xi \in \chi_0$, we have*

$$\left| \langle \nabla U(\xi),\, v \rangle_{L^2(d\xi)} \right| \leq \frac{u_{|\xi|}}{\operatorname{per}\, J(\xi)}. \tag{7.4.17}$$

An analog of Theorem 7.4.1 now becomes.

Theorem 7.4.5 *Assume that K is of finite rank N and that the kernel J is once differentiable with continuous derivative. Then, Hypothesis 4 is satisfied.*

Proof. Since K is of finite rank N there are at most N points in any configuration. It is clear from (7.4.16) that $(t \mapsto U(\phi_t^v(\xi)))$ is differentiable. Since $|\det J(\xi)[\tau]| \leq c|\tau|^{|\tau/2|}$ where $|\tau|$ is the cardinal of $\tau \in D[|\xi|]$, we get

$$\left| \langle \nabla U(\xi),\, v \rangle_{L^2(d\xi)} \right| \leq c\, \frac{\mathfrak{B}_{|\xi|} |\xi|^{|\xi|/2}}{\operatorname{per}\, J(\xi)} \mathbf{1}_{\{|\xi| \leq N\}}.$$

Hence the result. ■

Remark 7.4.2 *The finite rank condition is rather restrictive but the sequence $(\mathfrak{B}_n n^{n/2},\, n \geq 1)$ has not a finite exponential generating function thus we can't avoid it. In order to circumvent this difficulty one would have to improve known upper-bounds on permanents.*

We can then state the main result for this subsection.

Theorem 7.4.6 *Assume that $(E,\, K,\, \lambda)$ satisfy Hypothesis 1, 2 and 4. Let F and G belong to $\mathcal{F}C_b^\infty$. For any compact Λ, we have:*

$$\begin{aligned} \int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi)\ d\mu_{1,K_\Lambda,\lambda}(\xi) &= - \int_{\chi_\Lambda} F(\xi) \nabla_v G(\xi)\ d\mu_{1,K_\Lambda,\lambda}(\xi) \\ &\quad + \int_{\chi_\Lambda} F(\xi) G(\xi) \left(B_v^\lambda(\xi) + \nabla_v U(\xi) \right)\ d\mu_{1,K_\Lambda,\lambda}(\xi). \end{aligned}$$

Proof. Same as the proof of Theorem 7.4.2. ■ Now then, we can work as in Subsection 7.4.2 and we obtain the integration by parts formula for α-permanental point processes.

Corollary 2 *Assume that $(E_1,\, K_1, \lambda_1)$ satisfy Hypothesis 1, 2 and 4. Let $s = 1/\alpha$ be an integer greater than* 1*. For F and G cylindrical functions, for $v \in V_0(E_1)$, we have:*

$$\int_{\chi_\Lambda} \nabla_v F(\omega) G(\omega)\ d\mu_{\alpha,\, sK_{1,\Lambda},\lambda_1}(\omega) = -\int_{\chi_\Lambda} F(\omega)\nabla_v G(\omega)\ d\mu_{\alpha,\, K_{1,\,\Lambda},\lambda_1}(\omega)$$
$$+\frac{1}{\alpha}\int_{\chi_\Lambda} F(\omega)G(\omega)\left(\sum_{\eta\subset\omega}(B_v^{\lambda_1}(\eta)+\nabla_v U(\eta))R(\eta,\,\omega)\right)\ d\mu_{\alpha,\, sK_{1,\Lambda},\lambda_1}(\omega).$$

7.5 Conclusion

We showed that for any $\alpha \in \mathfrak{A}$, a stochastic integration by parts formula holds. A first well known consequence of such a formula is the closability of ∇. We define the norm $||.||_{2,1}$ on $\mathcal{F}C_b^\infty(\mathcal{D},\chi)$ by:

$$\begin{aligned} ||F||_{2,1}^2 &= ||F||_{L^2(\mu)}^2 + \mathbb{E}\left[||\nabla F||^2\right] \\ &= \mathbb{E}\left[F^2\right] + \mathbb{E}\left[\int |\nabla_x F|^2 \ \mathrm{d}\xi(x)\right]. \end{aligned}$$

and we call $\mathcal{D}_{2,1}$ the closure of $\mathcal{F}C_b^\infty(\mathcal{D},\chi)$ for the norm $||.||_{2,1}$. A classical consequence of the previous results is then that, for any α-DPPP known to exist, the operator ∇ is closable and can thus be extended to $\mathcal{D}_{2,1}$. Moreover, the integration by parts remains valid as is for F and G in $\mathcal{D}_{2,1}$. With the same lines of proof we retrieve the result of [45], which says that the Dirichlet form: $\mathcal{E}(F,F) = \mathbb{E}\left[\langle \nabla F, \nabla F\rangle\right]$ is closable.

Bibliography

[1] M. Aigner, *Combinatorial theory*, Classics in Mathematics, Springer-Verlag, Berlin, 1997, Reprint of the 1979 original.

[2] S. Albeverio, Y. G. Kondratiev, E. Lytvynov, and G. Ustev, *Analysis and geometry on marked configuration spaces*, Infinite dimensional harmonic analysis (Kyoto, 1999), Gräbner, Altendorf, 2000, pp. 1–39.

[3] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, *Analysis and geometry on configuration spaces*, J. Funct. Anal. **154** (1998), no. 2, 444–500.

[4] ———, *Analysis and geometry on configuration spaces: the Gibbsian case*, J. Funct. Anal. **157** (1998), no. 1, 242–291.

[5] V. Bally, M.-P. Bavouzet, and M. Messaoud, *Integration by parts formula for locally smooth laws and applications to sensitivity computations*, Ann. Appl. Probab. **17** (2007), no. 1, 33–66.

[6] R.F. Bass and M. Cranston, *The Malliavin calculus for pure jump processes and applications to local time*, The Annals of Probability **14** (1986), no. 2, 490–532.

[7] K. Bichteler and J. Jacod, *Calcul de Malliavin pour les diffusions avec sauts : existence d'une densité dans le cas unidimensionnel*, Séminaire de probabilités XVII, Springer-Verlag, New-York, 1983, pp. 132–157.

[8] J.-M. Bismut, *Calcul des variations stochastiques et processus de sauts*, Zeitschrift für Wahrscheinlichkeits Theory **63** (1983), 147–235.

[9] L.V. Bogachev and A. Daletskii, *Analysis on configurations spaces and Gibbs cluster ensembles*, Russian Journal of Mathematical Physics **14** (2007), no. 4, 397–408.

[10] N. Bouleau and L. Denis, *Energy image density property and the lent particle method for Poisson measures*, J. Funct. Anal. **257** (2009), no. 4, 1144–1174.

[11] A. Borodin and G. Olshanski, *Distributions on partitions, point processes and the hypergeometric kernel*, Comm. Math. Phys. **211** (2000) 355-358, [arXiv:math/9904010].

[12] A. Borodin, P. Diaconis, J. Fulman, *On adding a list of numbers (and other one-dependant determinantal processes*, (2009), Preprint.

[13] A. Borodin, *Determinantal point processes*, (2009).

[14] P. Bremaud, *Point Processes and Queues*, Martingale Dynamics, Springer.

[15] I. Camilier, L. Decreusefond, *Quasi-invariance and integration by parts for determinantal and permanental processes*, Journal of Functional Analysis, (2010), 259, 1, 260–300.

[16] W. Chu, *Determinant, permanent, and MacMahon's master theorem*, Linear Algebra and its Applications **255** (1997), no. 1-3, 171–183.

[17] D. J. Daley and D. Vere-Jones, *An introduction to the theory of point processes. Vol. I*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2003, Elementary theory and methods.

[18] L. Decreusefond and N. Savy, *Anticipative calculus with respect to filtered Poisson processes*, Annales de l'Institut Henri Poincaré (B) Probability and Statistics **42** (2006), no. 3, 343–372.

[19] G. Di Nunno, B. Oksendal, and F. Proske, *White noise analysis for Lévy processes*, Journal of Functional Analysis **206** (2004), no. 1, 109–148.

[20] N. Dunford and J.T. Schwartz, *Linear operators. part ii*, Wiley Classics Library, 1988 (English).

[21] N. Eisenbaum, H. Kaspi, *On Permanental Processes*, 2006

[22] H.O. Georgii and H.J. Yoo, *Conditional intensity and Gibbsianness of determinantal point processes*, J. Stat. Phys., (2005) 118(1-2):55-84.

[23] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Determinantal processes and independence*, Probab. Surv. **3** (2006), 206–229 (electronic).

[24] ———, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, (2007).

[25] K. Johansson, *Toeplitz determinants, random growth and determinantal processes*, Proceedings of the ICM, vol. 3, 2002, pp. 53–62.

[26] ______, *Discrete polynuclear growth and determinantal processes*, Comm. Math. Phys. **242** (2003), no. 1-2, 277–329.

[27] ______, *Random matrices and determinantal processes*, math-ph (2005), no. 0510038.

[28] O. Kallenberg, *Random measures*, 3rd ed., Academic Press, 1983.

[29] M. Katori and H. Tanemura, *Noncolliding processses, matrix-valued processes and determinantal processes*, (2010), [arXiv:1005.0533v1].

[30] R. Lyons, *Determiantal probability measures*, Publ. Math. Inst. Hautes Etudes Sci. **98** (2003) 167-212, [arXiv:math/0204325].

[31] O. Macchi, *The coincidence approach to stochastic point processes*, Advances in Appl. Probability **7** (1975), 83–122.

[32] P. McCullagh, J. Moller, *The permanental process*, Advances in applied probability, 38, 873 − 888.

[33] J. Picard, *Formules de dualité sur l'espace de Poisson*, Ann. Inst. H. Poincaré Probab. Statist., 1996, no. 4, vol. 32, 509–548.

[34] N. Privault, *A transfer principle from Wiener to Poisson space and applications*, Journal of Functional Analysis, 1995, no. 2, vol. 132,

[35] ______, *Stochastic analysis in discrete and continuous settings with normal martingales*, Lecture Notes in Mathematics, vol. 1982, Springer-Verlag, Berlin, 2009.

[36] T. Shirai, *Remarks on the positivity of alpha-determinants*, Vol.**61**, 2007, 169-189.

[37] T. Shirai and Y. Takahashi, *Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes*, J. Funct. Anal. **205** 2003, no. 2, 414–463.

[38] B. Simon, *Trace ideals and their applications*, second ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.

[39] A. Soshnikov, *Determinantal random point fields*, Russian Mathematical Surveys **55** 2000, no. 5, 923–975.

[40] ———, *Determinantal Random Fields*, in: Encyclopedia of Mathematical Physics, pp. 47-53, Oxford: Elsevier, 2006.

[41] H. Spohn, *Hydrodynamic behavior and interacting particle systems*, (1987), Springer.

[42] H. Tamura and K. R. Ito, *A canonical ensemble approach to the fermion/boson random point processes and its applications*, Communications in Mathematical Physics **263** (2006), 353.

[43] A.S. Üstünel and M. Zakai, *Transformations of measure on Wiener spaces*, Monographs in Mathematics, Springer-Verlag, 2000.

[44] D. Vere-Jones, *Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions*, New Zealand Journal of Mathematics, Volume **26**, 1997, 125-149.

[45] H. J. Yoo, *Gibbsianness of fermion random point fields*, Math. Z. **252** (2006), no. 1, 27–48.

[46] H.J. Yoo, *Dirichlet forms and diffusion processes for fermion random point fields*, J. Funct. Anal. **219** (2005), no. 1, 143–160.

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